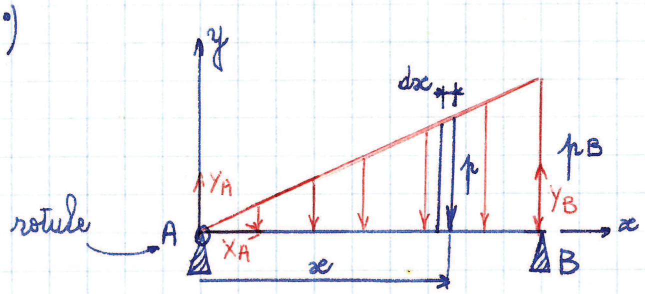
# TD RDM

EX: Déterminer les actions inconnues

1°)




Bilan : $X_A$, $Y_A$ et $Y_B$

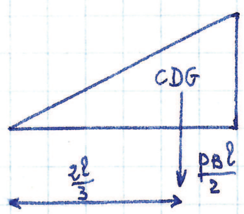
statique $X_A = 0$ $\qquad p = p_B \cdot \frac{x}{\ell}$

$$Y_A + Y_B + \int_0^\ell - p\,dx = 0$$

$$\rightarrow \quad Y_A + Y_B - \frac{p_B}{\ell}\left[\frac{x^2}{2}\right]_0^\ell = 0 = Y_A + Y_B - \frac{p_B\,\ell}{2} = 0$$

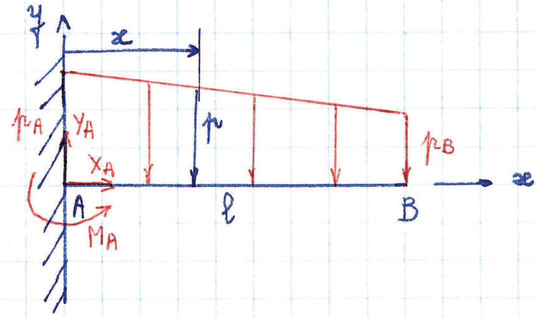
$$\Sigma\, \mathcal{M}t/A = 0 \Rightarrow \quad \ell\, Y_B - \int_0^\ell x\, p\,dx = 0$$

$$\ell\, Y_B - \frac{p_B}{\ell}\int_0^\ell x^2\,dx = 0 \Rightarrow \ell\, Y_B - \frac{p_B\,\ell^2}{3} = 0$$




$$\rightarrow \begin{cases} Y_B = \frac{p_B\,\ell}{3} \\ Y_A = \frac{p_B\,\ell}{6} \\ X_A = 0 \end{cases}$$

2°) calculer les réactions inconnues.




Bilan $X_A$, $Y_A$, C

statique : $X_A = 0$

$$Y_A + \int_0^\ell p(x)\,dx = 0 \qquad p(x) = ax + b \quad \text{(droite)}$$

pour $x = 0$ $p = p_A$ pour $x = \ell$ $p = p_B$

$$\Rightarrow \left.\begin{array}{l} b = p_A \\ p_B = a l + p_A \end{array}\right\} \Rightarrow a = \frac{p_B - p_A}{l}$$

$$\Rightarrow p(x) = \frac{p_B - p_A}{l} x + p_A$$

$$Y_A - \int_0^l dx \left( \frac{p_B - p_A}{l} x + p_A \right) = 0 \Rightarrow Y_A - \left[ \frac{p_B - p_A}{l} \frac{x^2}{2} + p_A x \right]_0^l = 0$$

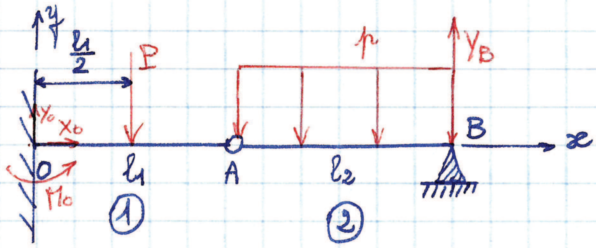
$$Y_A = \frac{p_A + p_B}{2} \times l$$

$$\Sigma \mathcal{M}_{F/A} = 0 \qquad M_A - \int_0^l p(x)\, dx \cdot x = 0$$

$$\Rightarrow M_A = \int_0^l \frac{P_B - P_A}{l} x^2\, dx + \int_0^l p_A x\, dx$$

$$M_A = \frac{l^2}{6} (2 P_B + P_A)$$

3°)




Calculer les actions inconnues.

Bilan : $X_0, Y_0, M_0, Y_B$

statique à ① + ② : $X_0 = 0$

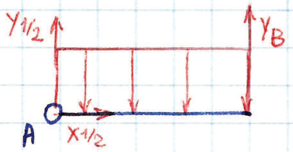
$$Y_B + Y_0 - P - p l_2 = 0$$

$$\mathcal{M}_{F/O} = -\frac{P l_1}{2} + Y_B (l_1 + l_2) - \left(l_1 + \frac{l_2}{2}\right) p l_2 = 0 - M_0$$

supposons que $l_1 = l_2 = l$

$$\Rightarrow \left\{\begin{array}{l} Y_0 + Y_B - p l - P = 0 \\ M_0 - \frac{P l}{2} + 2 l Y_B - \frac{3}{2} p l^2 = 0 \end{array}\right. \qquad \left\{\begin{array}{l} 2 \text{ équations.} \\ 3 \text{ inconnues.} \end{array}\right.$$

Isolons ②



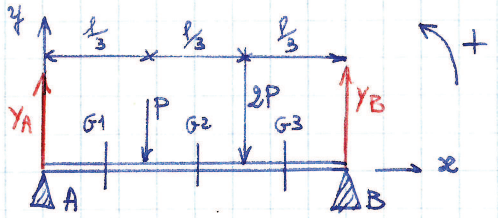

$$\Sigma \mathcal{M}_{/A} = 0 = Y_B l - \frac{p l^2}{2} = 0$$

$$\rightarrow Y_B = \frac{p l}{2}$$

$$\Rightarrow \left\{\begin{array}{l} Y_0 = P + \frac{p l}{2} \\ M_0 = \frac{1}{2} p l^2 + \frac{P l}{2} \end{array}\right.$$

$/\vec{x}$ $\begin{cases} X_{1/2} = 0 \\ Y_{1/2} = -Y_B + pl \end{cases} \Rightarrow Y_{12} = \frac{pl}{2}$

1°)




a) Déterminez le torseur des efforts intérieur en ∀ point

b) Tracez les diagrammes de $M_3$ et $T_2$.

$$\mathcal{M}^t/A = l Y_B - \frac{Pl}{3} - 2P\frac{l\times 2}{3} = 0$$

$$\begin{cases} Y_B = \frac{5}{3}P \\ Y_A = \frac{4}{3}P \end{cases}$$

1ère méthode.

$0 < x < \frac{l}{3}$ $\quad T_2 = -Y_A = -\frac{4}{3}P = (-F_G) = (+F_{dte})$

$M_3 = -\left(-\frac{4}{3}Px\right) = \frac{4}{3}Px$

$\frac{l}{3} < x < \frac{2l}{3}$ $\quad T_2 = -2P + \frac{5}{3}P = -\frac{P}{3}$

$M_3 = -\left(\frac{2l}{3} - x\right)2P + (l - x)\frac{5}{3}P = \frac{P}{3}(l + x)$

$\frac{2l}{3} < x < l$ $\quad T_2 = \frac{5}{3}P$

$M_3 = (l - x)\frac{5}{3}P.$

2e méthode. Equations locales de l'équilibre.

$(ds = dx)$ $\begin{cases} \frac{dM_3}{dx} + T_2 = 0 \\ \frac{dT_2}{dx} + p_2 = 0 \end{cases}$ si poutre droite.

ici $p_2 = 0$ (pas de charges réparties). $\rightarrow \frac{dT_2}{dx} = 0 \rightarrow T_2 = c^{te}$.

moment fléchissant

$0 < x < \frac{l}{3}$ $\quad \frac{dM_3}{dx} = -T_2 = \frac{4P}{3} \rightarrow dM_3 = \frac{4}{3}P\,dx$

$\rightarrow M_3 = \frac{4}{3}Px + C1$ (constante 1) $x = 0$ $M = 0 \rightarrow C1 = 0.$

$\frac{l}{3} < x < \frac{2l}{3}$ $\quad \frac{dM_3}{dx} = \frac{P}{3} \Rightarrow M_3 = \frac{P}{3}x + C_2$ $\quad$ pour $x = \frac{2l}{3}$ $\quad$ TRDM4

$M_3\left(\frac{2l}{3}\right) = \frac{5Pl}{9}$ (m^t $Y_B$)

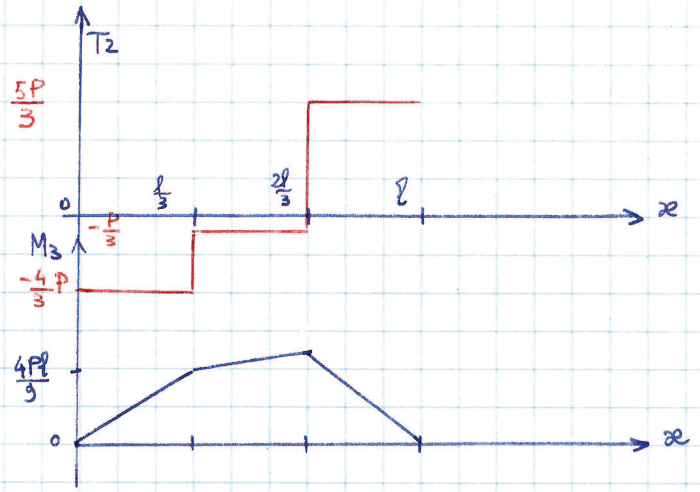
$$\Rightarrow \frac{5Pl}{9} = \frac{P}{3}\underbrace{x}_{=\frac{2l}{3}} + C_2 \rightarrow C_2 = \frac{1}{3}Pl$$

$$M_3 = \frac{P}{3}(x+l)$$

$\frac{2l}{3} < x < l$ $\quad \frac{dM_3}{dx} = -\frac{5}{3}P \rightarrow M_3 = -\frac{5}{3}Px + C_3 \rightarrow C_3 = \frac{5}{3}Pl \quad (x = l \quad M = 0)$

$$\rightarrow M_3 = \frac{5}{3}P(l - x)$$

Diagramme.

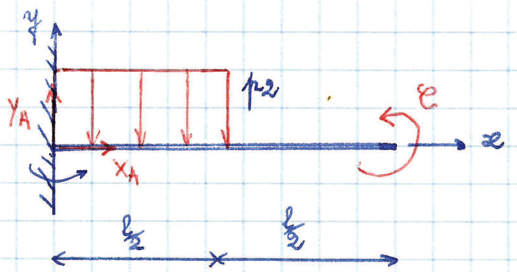



$\frac{dM_3}{dx} = -T_2$

discontinuité de T

$\rightarrow$ chgt de pente sur le diagramme de M

1°)




1) Torseur des efforts intérieurs.

2) Diagrammes

statique $\rightarrow X_A = 0$

$$\begin{cases} -p_2\frac{l}{2} + Y_A = 0 & \Rightarrow Y_A = p_2\frac{l}{2} \\ M_A + \mathcal{C} - p_2\frac{l^2}{8} = 0 & M_A = p_2\frac{l^2}{8} - \mathcal{C} \end{cases}$$

$x \in (0, \frac{l}{2})$ $\quad T_2 = -\left[\overbrace{\frac{p_2 l}{2}}^{Y_A} - p_2 x\right] = p_2\left(x - \frac{l}{2}\right)$

$$M_3 = -\left[\frac{p_2 l^2}{8} - \mathcal{C} + \frac{p_2 x^2}{2} - \frac{p_2 l x}{2}\right] = -\frac{p_2}{2}\left(x + \frac{l}{2}\right)^2 + \mathcal{C}$$

$x \in (\frac{l}{2}, l)$ $\quad T_2 = 0$

2^e méthode

$x \in (0, \frac{\ell}{2})$ : $\frac{dT_2}{dx} + p_2 = 0$

ici $p_2 = -p_2 \vec{y}$

$\frac{dT_2}{dx} = p_2$

$T_2 = p_2 x + C_1$

$x = \frac{\ell}{2}$ $T_2 = 0 \rightarrow C_1 = -\frac{p_2 \ell}{2}$

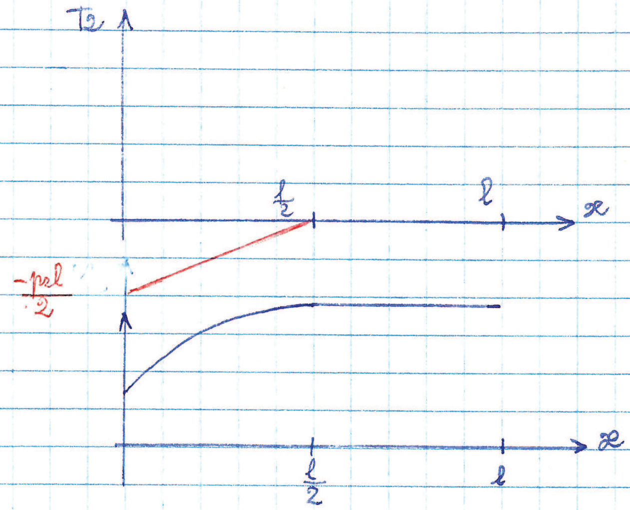
$\rightarrow T_2 = p_2 (x - \frac{\ell}{2})$

$\frac{dM_3}{dx} = -p_2 (x - \frac{\ell}{2}) \Longrightarrow M_3 = -p_2 \frac{1}{2} (x - \frac{\ell}{2})^2 + C_2$

pour $x = \frac{\ell}{2}$ $M_3 = \mathcal{C}$

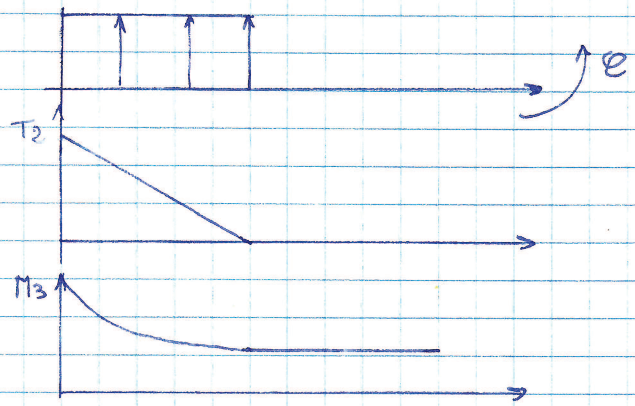
$\Longrightarrow C_2 = \mathcal{C}$

$\Longrightarrow M_3 = -\frac{1}{2} p_2 (x - \frac{\ell}{2})^2 + \mathcal{C}$.

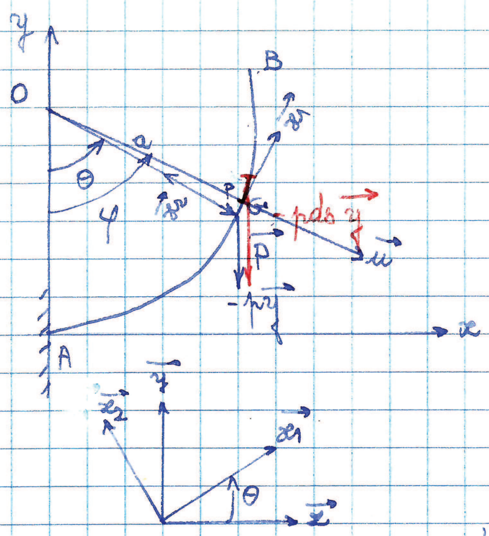



$\frac{dM_3}{dx} = -T_2 > 0$

(croissante $\rightarrow$ pente $> 0$)




1) I.




$$dN = -p\,ds\,\underbrace{\vec{y}\cdot\vec{x_1}}_{\sin\theta}$$

$$dN = -p\,ds\,\sin\theta$$

$$ds = a\,d\varphi$$

$$dN = -pa\,d\varphi\,\sin\theta$$

$$N = \int_\theta^{\frac{\pi}{2}} -pa\sin\theta\,d\varphi = -pa\sin\theta\left[\frac{\pi}{2}-\theta\right]$$

$$dT_2 = -p\,ds\,\underbrace{\vec{y}\cdot\vec{x_2}}_{\cos\theta} = -p\cos\theta\,a\,d\varphi$$

$$\rightarrow \quad T_2 = -pa\cos\theta\left[\frac{\pi}{2}-\theta\right]$$

$$\vec{dM_3} = \vec{GP}\wedge(-p\,ds\,\vec{y}) = (\vec{GO}+\vec{OP})\wedge(-pds\,\vec{y})$$

$$= (a\vec{x_2}+a\vec{u})\wedge -pds\,\vec{y}$$

$$= -apds\,(\vec{x_2}\wedge\vec{y}+\vec{u}\wedge\vec{y})$$

$$= -apds\,(-\sin\theta\,\vec{z}+\sin(\pi-\varphi)\,\vec{z})$$

$$\vec{dM_3} = -a^2p\cdot[-\sin\theta+\sin\varphi]\,d\varphi\,\vec{z}$$

$$\rightarrow \quad M_3 = -a^2p\int_\theta^{\frac{\pi}{2}}[-\sin\theta+\sin\varphi]\,d\varphi$$

$$= -a^2p\left[-\varphi\sin\theta-\cos\varphi\right]_\theta^{\frac{\pi}{2}} = -a^2p\left[-\frac{\pi}{2}\sin\theta+\theta\sin\theta+\cos\theta\right.$$

$$\Rightarrow \quad M_3 = -a^2p\left[\sin\theta\left(\theta-\frac{\pi}{2}\right)+\cos\theta\right]$$

2e méthode : Equation locale de l'équilibre.

$$\begin{cases} \dfrac{dN}{ds}+p_1-\dfrac{T_2}{R}=0 \\[2ex] \dfrac{N}{R}+\dfrac{dT_2}{ds}+p_2=0 \end{cases}$$

poutre plane non droite.

ici $\begin{cases} p_1 = -p\,\vec{y}\cdot\vec{x_1} = -p\sin\theta \\ p_2 = -p\vec{y}\cdot\vec{x_2} = -p\cos\theta \end{cases}$ $\quad\begin{cases} s = a\theta \\ ds = a\,d\theta \end{cases}$

D'où :

$$\begin{cases} \dfrac{dN}{a\,d\theta} - ap\sin\theta - \dfrac{T_2}{a} = 0 \quad (1) \\[2ex] \dfrac{N}{a} + \dfrac{dT_2}{a\,d\theta} - ap\cos\theta = 0 \quad (2) \end{cases}$$

$$(1)' \Rightarrow \frac{d^2N}{d\theta^2} - ap\cos\theta - \frac{dT_2}{d\theta} = 0$$

$$\frac{d^2N}{d\theta^2} - ap\cos\theta - \frac{dT_2}{d\theta} = 0 \rightarrow \frac{d^2N}{d\theta^2} - ap\cos\theta - ap\cos\theta + N = 0$$

$$\frac{d^2N}{d\theta^2} + N = 2ap\cos\theta$$

solution sans $2^{nd}$ membre $\quad N = A\cos\theta + B\sin\theta$

solution particulière $\quad N_0 = k\theta\sin\theta$

$$\frac{dN}{d\theta} = k(\sin\theta + \theta\cos\theta)$$

$$\frac{d^2N}{d\theta^2} = k(2\cos\theta - \theta\sin\theta)$$

$$N = A\cos\theta + B\sin\theta + ap\theta\sin\theta$$

$$\theta = \frac{\pi}{2} \quad N\left(\frac{\pi}{2}\right) = 0 \rightarrow B = -ap\frac{\pi}{2}$$

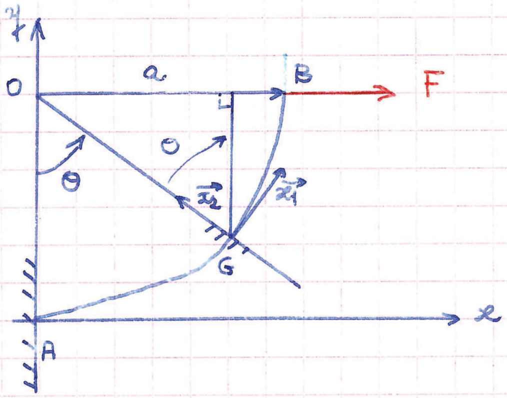
statique $\rightarrow \begin{cases} X_A = 0 \\ Y_A - \frac{\pi}{2}ap = 0 \end{cases}$

$$\theta = 0 \quad N(0) = 0 \rightarrow A = 0$$

D'où $\quad N = -ap\frac{\pi}{2}\sin\theta + ap\theta\sin\theta$

$$N = ap\sin\theta\left(\theta - \frac{\pi}{2}\right)$$

II-



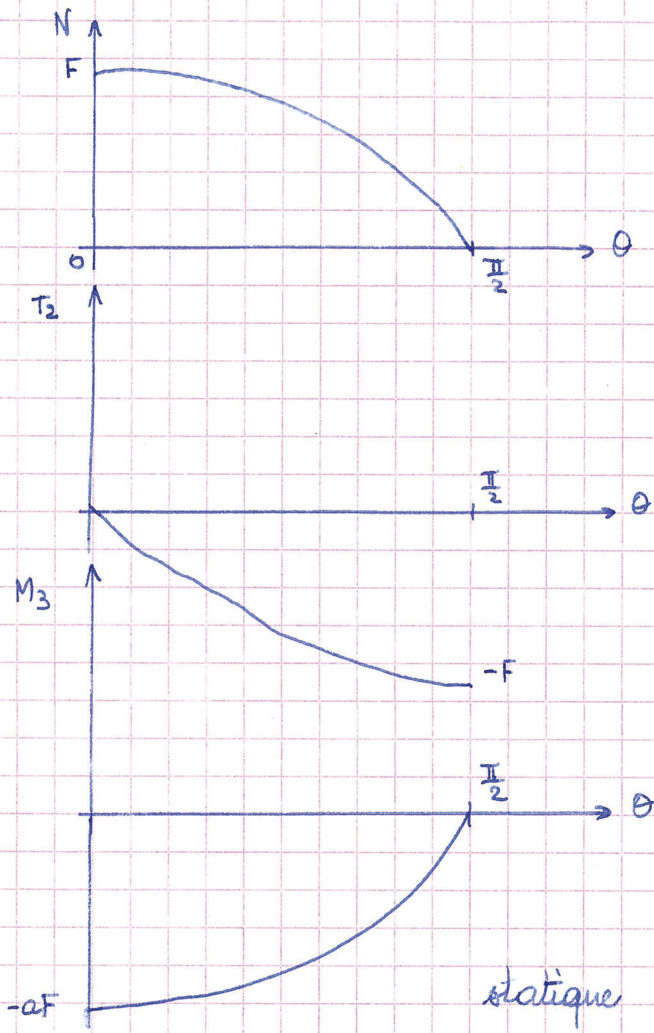

a) Déterminer $\begin{cases} N = N(\theta) \\ T_2 = T_2(\theta) \\ M_3 = M_3(\theta) \end{cases}$

b) diagrammes.

$$N = F\cdot\vec{x}\,\vec{x_1} = F\cos\theta$$

$$T_2 = F\underbrace{\vec{x}\cdot\vec{x_2}}_{-\sin\theta} = -F\sin\theta$$

$M_3 = \vec{GB} \wedge \vec{F} = -a\cos\theta \cdot F$




ici $p_1 = p_2 = 0$

$$\Rightarrow \begin{cases} \dfrac{dN}{d\theta} - T_2 = 0 & (\text{via } ds = a\,d\theta) \\ \dfrac{N}{a} + \dfrac{dT_2}{a\,d\theta} = 0 \end{cases}$$

$$\Rightarrow \frac{d^2N}{d\theta^2} - \underbrace{\frac{dT_2}{d\theta}}_{+N} = 0$$

$$N = -\frac{dT_2}{d\theta}$$

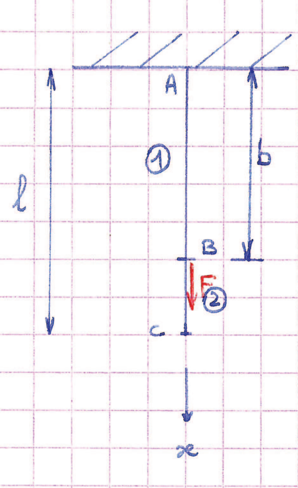
$$N = A\cos\theta + B\sin\theta$$

$\theta = \frac{\pi}{2}$ $N = 0 \rightarrow B = 0$

$\theta = 0$ $N = F \rightarrow A = F$

statique $\begin{cases} X_A + F = 0 \\ Y_A = 0 \end{cases}$

I -




un câble section S
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charges $F\vec{x}$ et pesanteur.

1) Calculer $N(x)$ ?

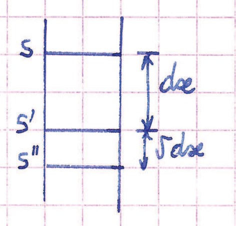
2) Déterminer $u(c)$

1) $x < b$

$N_1 = (l-x)\,S\rho g + F$

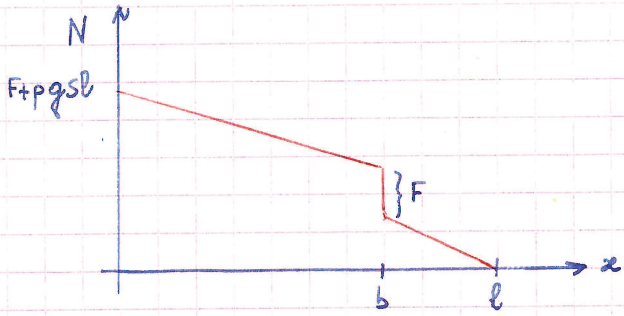
$b < x < l$ $N_2 = (l-x)\,S\rho g$

$$\varepsilon_{11}^{①} = \frac{1}{E}\,\frac{N_1}{S}$$




$$\frac{\delta_1 dx}{dx} = \varepsilon_{11}^{(1)}$$

$$\varepsilon_{11}^{①} dx = \delta_1\,dx = \frac{1}{ES}\left[(l-x)\rho S g + F\right]$$




$$u(B) = \frac{1}{ES}\int_0^b \left((l-x)\rho S g + F\right) dx = \frac{1}{ES}\left[-\rho S g \frac{(l-x)^2}{2} + Fx\right]_0^b$$

$$u(B) = \frac{1}{ES}\left[Fb + \rho S g \frac{l^2}{2} - \rho S g \frac{(l-b)^2}{2}\right]$$

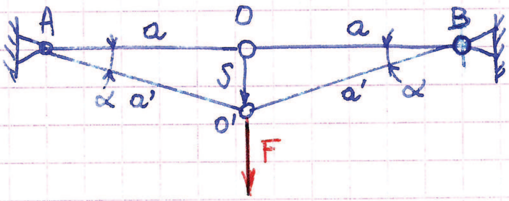
$$\varepsilon_{11}^{(2)} = \frac{N_2}{ES}$$

$$u(C) = u(B) + \int_b^l \varepsilon_{11}^{(2)} = u_B + \frac{1}{ES}\left[-\rho S g \frac{(l-x)^2}{2}\right]_b^l$$

$$u(C) = u(B) + \rho S g \frac{(l-b)^2}{2ES}$$

$$u_c = \frac{1}{ES}\left[Fb + \frac{\rho S g l^2}{2}\right] = \frac{1}{ES}\left[Fb + \frac{1}{2} Pl\right] \quad \text{vu} \quad P = \rho S l g$$

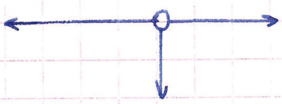
**EX 2 :**




E, S

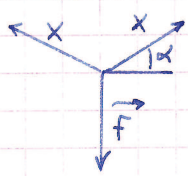
$\alpha$, $\delta$ petits

$\delta \ll a$ Déterminer $\delta$ en fonction de F.



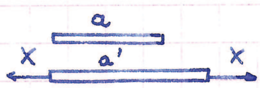
statique → équilibre impossible
les barres ne restent pas horizontales

⇓




$/\vec{x}$ $F - 2X\sin\alpha = 0$ $H = 1$ ;

$$\varepsilon_{11} = \frac{a'-a}{a} = \frac{X}{ES}$$




$a'$ : déformée

$$\Rightarrow \frac{a'}{a} = 1 + \frac{X}{ES} \quad ; \quad a'^2 = \delta^2 + a^2 \quad \text{(pythagore.)}$$

$$\left(\frac{a'}{a}\right)^2 = 1 + \frac{\delta^2}{a^2}$$

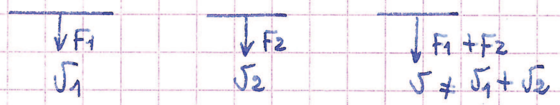
$$\frac{a'}{a} = \sqrt{1+\left(\frac{\delta}{a}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{\delta}{a}\right)^2$$

$$\Rightarrow \quad 1 + \frac{1}{2}\left(\frac{\delta}{a}\right)^2 = 1 + \frac{X}{ES}$$

$$/\vec{x} \quad \Rightarrow \quad X = \frac{F}{2\sin\alpha} \qquad \text{or} \quad \sin\alpha = \frac{\delta}{a'} = \frac{\delta}{a}$$

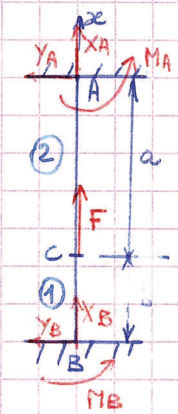
D'où : $$\frac{1}{2}\left(\frac{\delta}{a}\right)^2 = \frac{F}{2\sin\alpha}\,\frac{1}{ES}$$

$$\Rightarrow \left(\frac{\delta}{a}\right)^3 = \frac{F}{ES} \quad \Rightarrow \quad \delta = a\left(\frac{F}{ES}\right)^{\frac{1}{3}}$$




le principe de superposition ne marche pas.

EX 3 :




E, S

hyp : sollicitation de traction compression

1) Les actions inconnues.

2) diagramme $N(x)$.

3) Déplacement $u(c)$.

On suppose que la poutre travaille en traction compression

statique : $X_A + X_B + F = 0$

$H_e = 1$

Equation supplémentaire :

$x \in [0, b] \qquad N = -X_B \qquad \varepsilon_{11}^{(1)} = \frac{-X_B}{ES}$

$x \in [b, b+a] \qquad N = X_A \qquad \varepsilon_{11}^{(2)} = \frac{X_A}{ES}$

longueur $c^{te}$ de la poutre $\rightarrow \int_0^b \frac{-X_B}{ES}\,dx + \int_b^{b+a} \frac{X_A}{ES}\,dx = 0$

$\hookrightarrow \quad -X_B\, b + X_A\, a = 0 \Rightarrow X_A = X_B \frac{b}{a}$

$X_B + X_B \frac{b}{a} + F = 0$

$$\Rightarrow X_B = \frac{-aF}{a+b}$$

EX 1.

Ailette

Les sections sont homothétiques centre O .

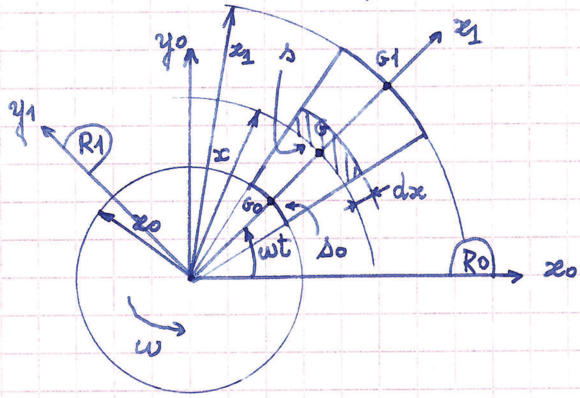
$\rho$ densité $E$ module de young

hyp : on néglige le poids

1) Etablir $N(x)$

2) Arbre indéformable, calculer $u(G1)$

3) Calculer le potentiel de l'ailette.




Equation locales de l'équilibre

$$\frac{dN}{dx_1} + p_1 = 0$$

└ composante sur $\vec{x_1}$ des efforts réparties / unité de longueur.

forces d'inertie d'entrainement

$$- dm \, \vec{\Gamma}_{G\,1/0} \qquad \text{avec} \qquad dm = \rho \underset{\substack{\uparrow \\ fct\,(x)}}{S} \, dx$$

homothetie $\Rightarrow \frac{s}{s_0} = \left(\frac{x}{x_0}\right)^2$ rapport surface = carré rapport lg

pour volume = cube

A $\omega = c^{te}$

$$\vec{\Gamma}(G\,1/0) = -\omega^2 x \vec{x_1}$$

$$-dm\, \Gamma(G\,1/0) = \rho \underbrace{S_0 \left(\frac{x}{x_0}\right)^2}_{S} \omega^2 x \vec{x_1} \, dx$$

$$p_1(\varepsilon) = \rho S_0 \frac{\omega^2}{x_0^2} x^3$$

force d'inertie de coriolis

$$- dm \, \vec{\Gamma_c}(G) = - dm \, 2 \omega \vec{z_0} \wedge \underset{\substack{\| \\ 0}}{\vec{V}_{G/R_1}} \quad (\text{régime établi})$$

$$= 0$$

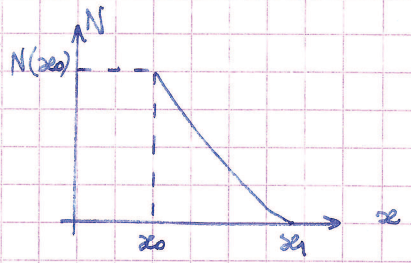
Donc $\frac{dN}{dx} + \rho S_0 \frac{\omega^2}{x_0^2} x^3 = 0$

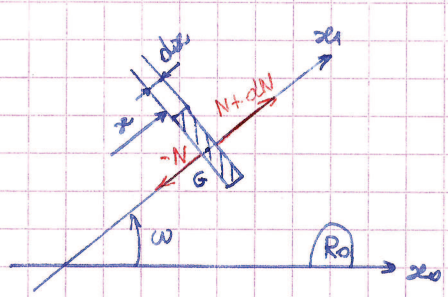
$$\Rightarrow dN = -\rho S_0 \frac{\omega^2}{x_0^2} x^3 \, dx$$

Intégrons $\quad N = -\rho\, s_0 \dfrac{\omega^2}{x_0^2} \dfrac{x^4}{4} + K$

pour $\quad x = x_1 \quad N = 0 \rightarrow K = \rho\, S_0 \dfrac{\omega^2}{x_0^2} \dfrac{x_1^4}{4}$

$$\Longrightarrow \quad N = \rho\, S_0 \frac{\omega^2}{4 x_0^2} \left( x_1^4 - x^4 \right)$$




$$dm\, \vec{\Gamma}_{G/0} = -N \vec{x_1} + (N + dN) \vec{x_1}$$

$$\rho S\, dx \left( -\omega^2 x \vec{x_1} \right) = dN \vec{x_1}$$

$$-\rho S \omega^2 x = \frac{dN}{dx}$$

2) Allongement de l'ailette

$$\sigma_{11} = \frac{N}{S} = \frac{\rho S_0 \omega^2}{4 x_0^2} \left( x_1^4 - x^4 \right) \cdot \frac{x_0^2}{s_0 x^2}$$

$$\Longrightarrow \quad \sigma_{11} = \frac{\rho \omega^2}{4} \left( \frac{x_1^4}{x^2} - x^2 \right)$$

On a $\quad \varepsilon_{11} = \dfrac{1}{E} \sigma_{11} = \dfrac{\rho \omega^2}{4E} \left( \dfrac{x_1^4}{x^2} - x^2 \right)$

$$u_1(G_1) = \int_{x_0}^{x_1} \varepsilon_{11}\, dx = \frac{\rho \omega^2}{4E} \int_{x_0}^{x_1} \left[ \frac{x_1^4}{x^2} - x^2 \right] dx$$

$$u_1(G_1) = \frac{\rho \omega^2}{4E} \left[ -\frac{x_1^4}{x^2} - \frac{x^3}{3} \right]_{x_0}^{x_1} = \frac{\rho \omega^2}{4E} \left[ -x_1^3 - \frac{x_1^3}{3} + \frac{x_1^4}{x_0} + \frac{x_0^3}{3} \right]$$

$$u_1(G_1) = \frac{\rho \omega^2}{4E} \left( -\frac{4}{3} x_1^3 + \frac{x_1^4}{x_0} + \frac{x_0^3}{3} \right)$$

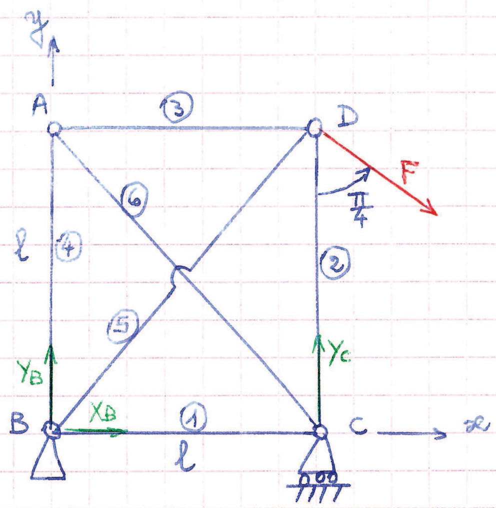
3) Potentiel

$$dW = \frac{1}{2} \frac{N^2}{ES} dx$$

$$W_{\text{ailette}} = \frac{1}{2} \int_{x_0}^{x_1} \frac{N^2}{ES} dx = \ldots\ldots$$

<u>EX 2 :</u>

barres : section S , E.




1) Degré d'hyperstaticité
2) Efforts dans les barres.
3) composantes $u_D$, $v_D$ du dplt du point D.

1ère étape : mise en place des inconnues auxiliaire (vert)

2e étape : statique

$$\begin{cases} X_B + F\frac{\sqrt2}{2} = 0 \\ Y_B + Y_C - F\frac{\sqrt2}{2} = 0 \\ m^t/B \quad Y_C\,l - l\sqrt2 F = 0 \end{cases} \Rightarrow \begin{cases} Y_C = \sqrt2 F \\ X_B = -F\frac{\sqrt2}{2} \\ Y_B = -F\frac{\sqrt2}{2} \end{cases}$$

On a tout pu calculer $\Rightarrow$ He $= 0$

3e étape . d° hyperstaticité intérieur

$$\begin{cases} b = 6 \\ n = 4 \end{cases}$$

$2n - 3 = 5$ équation indépts

b inconnues $= 6$ inconnues

$Hi = b - (2n-3) = 1$

$Hi = 1.$

Equation de l'équilibre des noeuds.

noeuds B . barres tendues.

$$\begin{cases} -F\frac{\sqrt2}{2} + N_1 + N_5\frac{\sqrt2}{2} = 0 & /x \\ -F\frac{\sqrt2}{2} + N_4 + N_5\frac{\sqrt2}{2} = 0 & /y \end{cases}$$

noeuds C

la barre tire sur le noeud C $\rightarrow$ (−)

$$\begin{cases} -N_1 - N_6\frac{\sqrt2}{2} = 0 & /x \\ \sqrt2 F + N_2 + N_6\frac{\sqrt2}{2} = 0 & /y \end{cases}$$

noeud D .

$$\begin{cases} -N_3 - N_5\frac{\sqrt2}{2} + F\frac{\sqrt2}{2} = 0 \\ -N_2 - N_5\frac{\sqrt2}{2} - F\frac{\sqrt2}{2} = 0 \end{cases}$$

<u>noeud A</u>
$$\begin{cases} N_3 + N_6 \frac{\sqrt{2}}{2} = 0 & /x \\ -N_4 - N_6 \frac{\sqrt{2}}{2} = 0 & /y \end{cases}$$

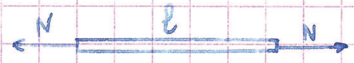
$H_e = 0 \quad H_i = 1 \longrightarrow$ 1 équation supplémentaire

1) th de Ménabréa.

Prenons une inconnue hyperstatique $N_5 = N$

si $W = W(1+2+3+4+5+6)$

$$\frac{\partial W}{\partial N} = 0$$



$$W = \frac{1}{2} \frac{N^2 \ell}{ES}$$

or $W = W_1 + W_2 + W_3 + W_4 + W_5 + W_6$

$$= \frac{\ell}{2ES} \left[ N_1^2 + N_2^2 + N_3^2 + N_4^2 + \boxed{\sqrt{2}} N_5^2 + \boxed{\sqrt{2}} N_6^2 \right]$$

équations aux noeuds $\Rightarrow$
$$\begin{cases} N_1 = \frac{\sqrt{2}}{2}(F-N) \longrightarrow \frac{\partial N_1}{\partial N} = -\frac{\sqrt{2}}{2} \\ N_4 = \frac{\sqrt{2}}{2}(F-N) \quad \frac{\partial N_4}{\partial N} = -\frac{\sqrt{2}}{2} \\ N_3 = \frac{\sqrt{2}}{2}(F-N) \quad \frac{\partial N_3}{\partial N} = -\frac{\sqrt{2}}{2} \\ N_2 = -\frac{\sqrt{2}}{2}(F+N) \quad \frac{\partial N_2}{\partial N} = -\frac{\sqrt{2}}{2} \\ N_6 = -\sqrt{2} N_1 = -(F-N) = N-F \longrightarrow \frac{\partial N_6}{\partial N} = 1 \end{cases}$$

vu $\frac{\partial W}{\partial N} = \frac{2\ell}{2ES} \left[ N_1 \frac{\partial N_1}{\partial N} + N_2 \frac{\partial N_2}{\partial N} + N_3 \frac{\partial N_3}{\partial N} + N_4 \frac{\partial N_4}{\partial N} + \sqrt{2} N + \sqrt{2} N_6 \frac{\partial N_6}{\partial N} \right]$

On a $\frac{\partial W}{\partial N} = 0 = -\left(\frac{\sqrt{2}}{2}\right)^2 (F-N) + \left(\frac{\sqrt{2}}{2}\right)^2 (F+N) - \left(\frac{\sqrt{2}}{2}\right)^2 (F-N) - \left(\frac{\sqrt{2}}{2}\right)^2 (F-N) + 2\sqrt{2} N$

$+ 2\sqrt{2}(N-F) = 0$

En simplifiant $\longrightarrow$ $N_5 = \frac{F}{2}$ (tendue)

$N_1 = \frac{\sqrt{2}}{4} F$ tendue

$N_2 = \frac{-3\sqrt{2}}{4} F$ comprimée

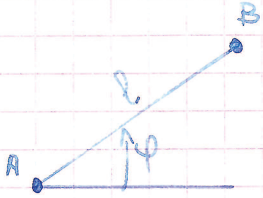
$N_3 = \frac{\sqrt{2}}{4} F$ tendue

$N_4 = \frac{\sqrt{2}}{4} F$ tendue

$N_5 = \frac{F}{2}$ tendue

$N_6 = -\frac{F}{2}$ comprimée

## 2^e méthode compatibilité des déplacements




$$(u_B - u_A)\cos\varphi + (v_B - v_A)\sin\varphi = \frac{Nl}{ES}$$

8 équations statiques

$$\begin{cases} u_B = 0 \\ v_B = 0 \end{cases} \quad \begin{cases} u_C \\ V_C = 0 \end{cases} \quad \begin{cases} u_A \\ v_A \end{cases} \quad \begin{cases} u_D \\ v_D \end{cases}$$

* barre ①. $\varphi_1 = 0$

$$(u_C - 0) = \frac{N_1 l}{ES}$$

* barre ②. $(u_D - u_C)\underset{\overset{\parallel}{0}}{\cos 90°} + v_D = \frac{N_2 l}{ES}$

* barre ③ $(u_D - u_A) = \frac{N_3 l}{ES}$

* barre ④ $v_A = \frac{N_4 l}{ES}$

* barre ⑤ $u_D \frac{\sqrt{2}}{2} + v_D \frac{\sqrt{2}}{2} = \frac{N_5 l \sqrt{2}}{ES}$

* barre ⑥

$$-(u_A - u_C)\frac{\sqrt{2}}{2} + (v_A - v_C)\frac{\sqrt{2}}{2} = \frac{N_6 l \sqrt{2}}{ES}$$

5 inconnues de déplacements $u_C, v_D, u_D, u_A, v_A$

nombre équations = 8 (statique) + 6 (compatibilité).

nombres inconnues = 6 + 5 = 11

→ résolution

2) Déplacement en D.

On a $v_D = \frac{N_2 l}{ES}$ (équation compatibilité barre ②)

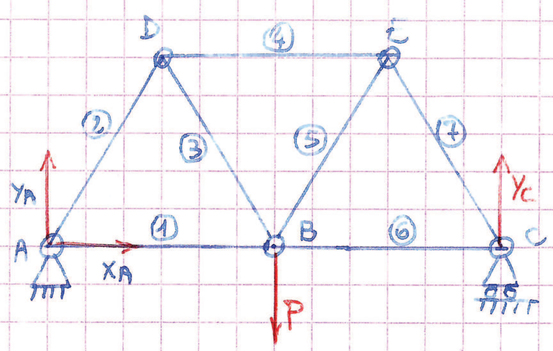
$$\Rightarrow v_D = -\frac{3\sqrt{2}}{4} F \frac{l}{ES} \quad \text{vu} \quad N_2 = -\frac{3\sqrt{2}}{4} F$$

⑤ → $u_D = \frac{Fl}{ES}\left(1 + \frac{3\sqrt{2}}{4}\right)$

De même ① ⇒ $u_C = \frac{N_1 l}{ES} = \frac{\sqrt{2}}{4}\frac{Fl}{ES}$

# EX 2






statique $\quad X_A = 0$

$$\left.\begin{array}{l} X_A = 0 \\ Y_C = Y_A = \frac{P}{2} \end{array}\right\} \Rightarrow H_e = 0$$

Equations supplementaires

intérieurement

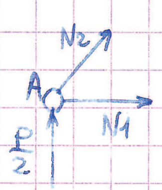
$b = 7 \rightarrow 7$ inc

$n = 5 \longrightarrow 10 - 3 = 7$ éq

$H_i = 0$ isostatique

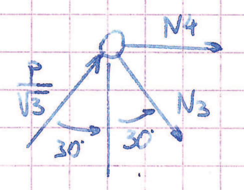
1ère méthode

Analytique (actions / noeud)




$$\begin{cases} N_1 + \frac{N_2}{2} = 0 \\ \frac{P}{2} + N_2 \frac{\sqrt{3}}{2} = 0 \end{cases} \Rightarrow \begin{cases} N_1 = \frac{P}{2\sqrt{3}} \\ N_2 = \frac{-P}{\sqrt{3}} \end{cases}$$

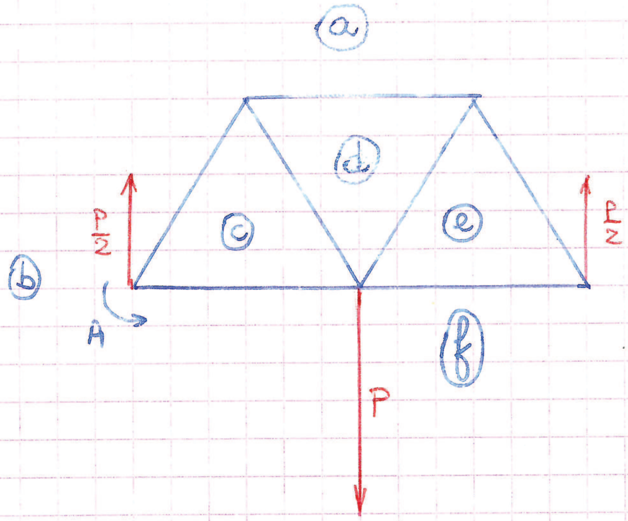
en D




$$\begin{cases} N_4 + \frac{N_3}{2} + \frac{P}{\sqrt{3}}\,\frac{1}{2} = 0 \\ -N_3 \frac{\sqrt{3}}{2} + \frac{P}{\sqrt{3}}\,\frac{\sqrt{3}}{2} = 0 \end{cases} \Rightarrow \begin{cases} N_3 = \frac{P}{\sqrt{3}} \\ N_4 = \frac{-P}{\sqrt{3}} \end{cases}$$

En refait pareil à tt les noeuds.

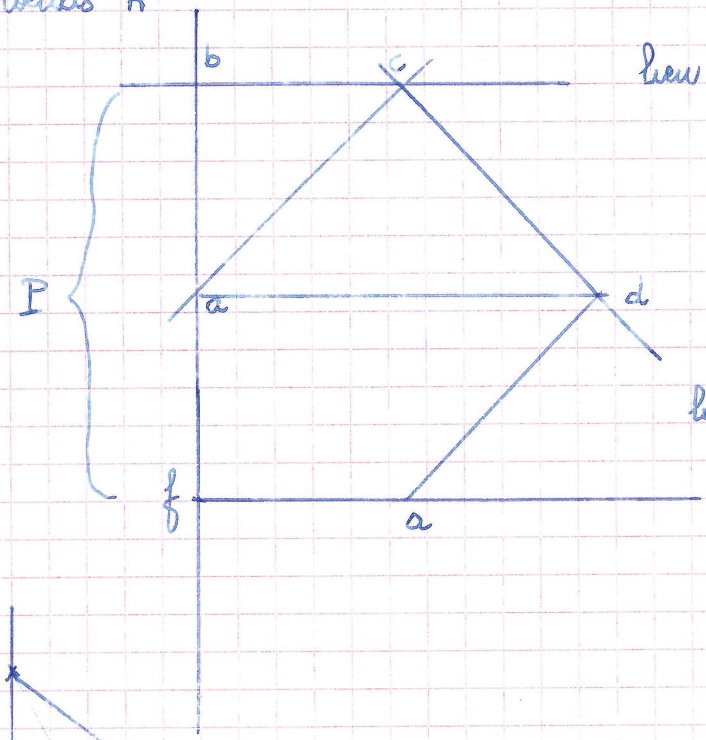
2° méthode : graphique




tourner sens trigo.

choisir une échelle de force.

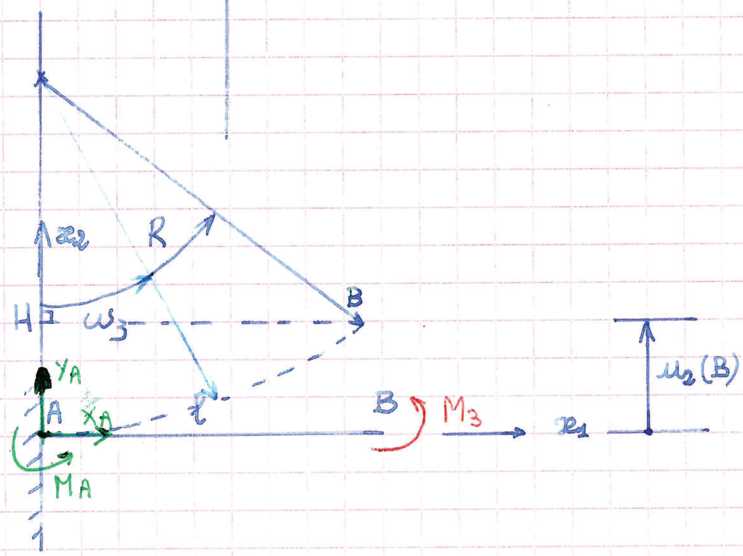
noeuds A




lieu de c

lieu de d

EX :




Déterminer la flèche $u_2(B)$ par

- flexion pure
- Equation différentielle de la déformée.

$$\begin{cases} X_A = 0 \\ Y_A = 0 \\ M_A + \underset{\overset{\shortparallel}{M_3}}{M_B} = 0 \end{cases} \rightarrow \quad M_A = -M_3 \quad \text{flexion pure.}$$

$$Mf_3 = M_3 = c^{te}.$$

En flexion pure $\frac{1}{R} = \frac{M_3}{EI} \rightarrow R = c^{te} \Rightarrow$ déformée = cercle.

On a $\ell = \omega_3 R$

$$\Rightarrow \omega_3 = \frac{\ell}{R} = \frac{M_3 \ell}{EI}$$

$$\Rightarrow \omega_3(B) = \frac{M_3 \ell}{EI}$$

flèche = $AH = R - R\cos\omega_3$

$$= R(1 - \cos\omega_3)$$

les déformations restent petites $\Rightarrow \omega_3$ petit

$$\Rightarrow AH \simeq R\left[1 - 1 + \frac{\omega_3^2}{2}\right] = \frac{R}{2}\omega_3^2 = \frac{R}{2}\frac{\ell^2}{R^2} = \frac{\ell^2}{2R}$$

$AH = u_2(B) = \frac{1}{2}\frac{M_3 \ell^2}{EI_3}$ (possible que si flexion pure)

2^e méthode : éq différentielle de la déformée

$$\frac{d^2 u_2}{dx_1^2} = \frac{Mf_3}{EI_3} = \frac{M_3}{EI_3}$$

Intégrons $\frac{du_2}{dx} = \frac{M_3}{EI_3} x_1 + C_1$

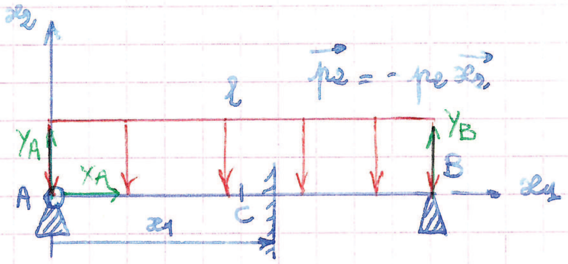
$$u_2 = \frac{M_3}{EI_3}\frac{x_1^2}{2} + C_1 x_1 + C_2$$

<u>CL</u> $x_1 = 0$

$$\left.\begin{array}{l} u_2 = 0 \rightarrow C_2 = 0 \\ \frac{du_2}{dx_1}(0) = 0 \rightarrow C_1 = 0 \end{array}\right\} \Rightarrow u_2(x_1) = \frac{M_3}{EI_3}\frac{x_1^2}{2}$$

flèche en B : $u_2(B) = \frac{M_3 \ell^2}{2EI_3}$

<u>EX.</u>




1) Déterminer l'équation différentielle de la déformée.

2) Calculer $u_2(C)$ et $\omega_3(B)$.

1° : inconnus de liaisons.

statique : $\left\{\begin{array}{l} X_A = 0 \\ Y_A + Y_B - P_2\ell = 0 \\ Y_A = Y_B \quad (\text{symétrie}) \end{array}\right\} \Rightarrow Y_A = Y_B = \dfrac{P_2\ell}{2}$

$$M^t_A = Y_B\,\ell - P_2\frac{\ell^2}{2} = 0 \Rightarrow Y_B = \frac{P_2\ell}{2}$$

$$M_3 = \frac{P_2\ell}{2}(\ell - x) - \frac{P_2(\ell-x)^2}{2}$$

$$EI_3\frac{d^2u_2}{dx_1^2} = \frac{P_2\ell}{2}(\ell-x) - \frac{P_2(\ell-x)^2}{2}$$

(A) $$EI_3\frac{du_2}{dx_1} = -\frac{P_2\ell}{4}(\ell-x)^2 + \frac{P_2}{6}(\ell-x)^3 + C_1$$

$$EI_3\,u_2 = \frac{P_2\ell}{12}(\ell-x)^3 - \frac{P_2}{24}(\ell-x)^4 + C_1x + C_2$$

Pour $\left\{\begin{array}{lll} x = \ell & u_2 = 0 & \Rightarrow C_1\ell + C_2 = 0 \\ x = 0 & u_2 = 0 & \dfrac{P_2\ell^4}{12} - \dfrac{P_2\ell^4}{24} = 0 - C_2 \end{array}\right.$

$$\Rightarrow C_2 = \frac{-P_2\ell^4}{24} \quad \text{et} \quad C_1 = \frac{P_2\ell^3}{24}$$

$$u_2(C) = \frac{P_2\ell^4}{EI_3}\left[\frac{1}{12}\cdot\frac{1}{8} - \frac{1}{24}\cdot\frac{1}{16} + \frac{1}{48} - \frac{1}{24}\right] = \frac{-5P_2\ell^4}{384\,EI_3} < 0.$$

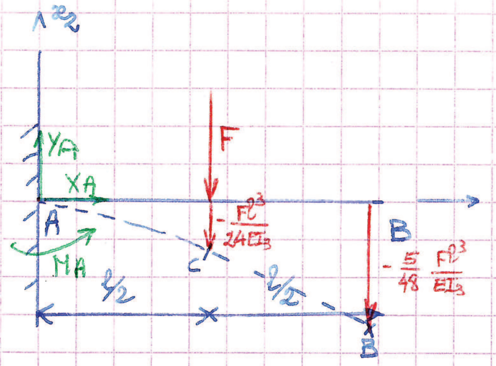
$$\omega_3(B) = \frac{C_1}{EI_3} \quad \text{vu (A)} \Rightarrow \omega_3(B) = \frac{P_2\ell^3}{24\,EI_3} > 0.$$

EX :

1) Equation différentielle de la déformée.

TRDM 2

2) $u_2(B)$ , $\omega_3(B)$




$$\begin{cases} X_A = 0 \\ Y_A - F = 0 \\ M_A - \frac{Fl}{2} = 0 \end{cases} \qquad \begin{cases} Y_A = F \\ M_A = \frac{Fl}{2} \end{cases}$$

$x \in \left]0, \frac{l}{2}\right[ \qquad M_3 = -F\left(\frac{l}{2} - x_1\right)$

$$EI_3 \frac{d^2u_2}{dx_1^2} = -F\left(\frac{l}{2} - x_1\right)$$

$$\Rightarrow EI_3 \frac{du_2}{dx_1} = \frac{F}{2}\left(\frac{l}{2} - x_1\right)^2 + C_1$$

$$EI_3\, u_2 = -\frac{F}{6}\left(\frac{l}{2} - x_1\right)^3 + C_1 x_1 + C_2$$

$$x_1 = 0 \quad u_2 = 0 \rightarrow \begin{cases} C_2 = \frac{Fl^3}{48} \\ C_1 = \frac{-Fl^2}{8} \end{cases}$$

$$u_2^{(1)} = \frac{F}{2EI_3}\left[-\frac{1}{3}\left(\frac{l}{2} - x_1\right)^3 - \frac{l^2}{4}x_1 + \frac{l^3}{24}\right]$$

$x_1 \in \left]\frac{l}{2}, l\right[ \qquad M_3 = 0$

$$EI_3 \frac{d^2u_2^{(2)}}{dx_1^2} = 0 \Rightarrow u_2^{(2)} = ax_1 + b$$

(2) ne se déforme pas

pour $x_1 = \frac{l}{2}$
$$\begin{cases} u_2^{(1)} = \frac{F}{2EI_3}\left(-\frac{l^3}{8} + \frac{l^3}{24}\right) = \frac{-Fl^3}{24EI_3} \\ \omega_3(C) = \frac{-Fl^2}{8EI_3} \end{cases}$$

$$u_2^{(2)}\left(\frac{l}{2}\right) = u_2^{(1)}\left(\frac{l}{2}\right) \Rightarrow a\frac{l}{2} + b = \frac{-Fl^3}{24EI_3}$$

$$\omega_3^{(2)}\left(\frac{l}{2}\right) = \omega_3^{(1)}\left(\frac{l}{2}\right) \rightarrow a = \frac{-Fl^2}{\ldots}$$

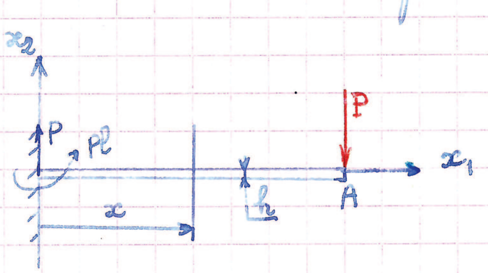
$$b = \frac{Fl^3}{8EI_3}\left(\frac{1}{6}\right) = \frac{Fl^3}{48EI}$$

$$u_2^{(2)} = \frac{-Fl^2}{8EI_3} x_1 + \frac{Fl^3}{48EI_3}$$

$\omega_3(C) = \omega_3(B).$

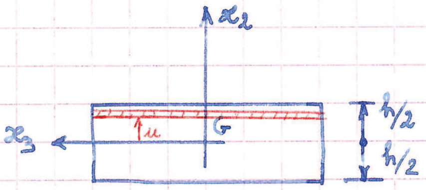
## Ex 1.

trouver la loi de variation $b = b(x_1)$ en largeur d'une pièce d'égale résistance à la flexion de hauteur $h$ constante.




$$\begin{cases} T_2 = -P \\ M_3 = -P(l - x_1) \end{cases}$$

$$\sigma = \frac{+P(l-x_1)\,h \times 12}{2\; bh^3} = Re \quad \text{vu} \quad \sigma_{11} = -\frac{M_3}{I_3} x_2$$

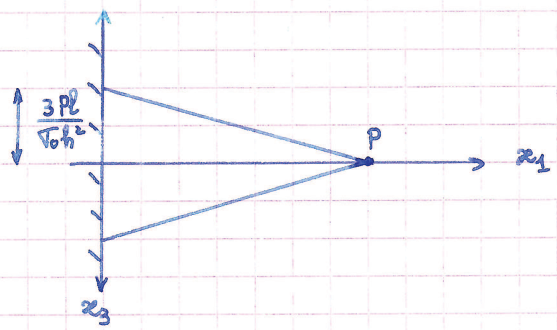



$$I_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} u^2 \underbrace{b\, du}_{S} = \frac{bh^3}{12}$$

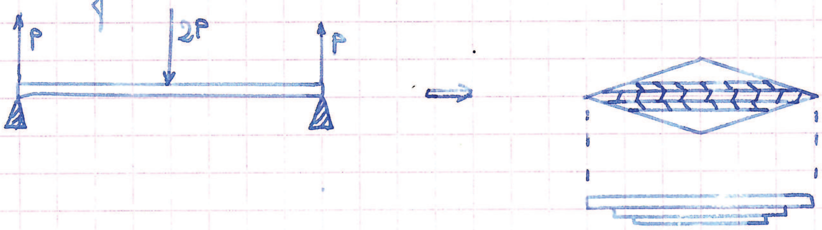
Contrainte maxi pour $x_2 = \pm \frac{h}{2}$

$$\sigma_{11}\ \text{maxi} = \sigma_0 = \frac{6P(l-x_1)}{bh^2}$$

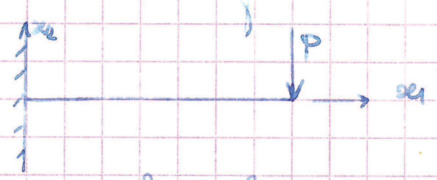
$$\Rightarrow \quad b = \frac{6P}{\sigma_0 h^2}(l - x_1)$$




Par analogie :

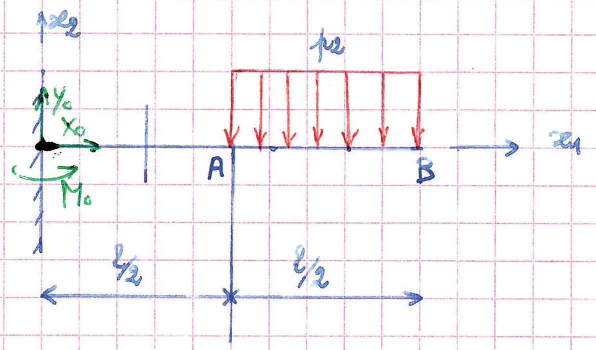


EX 2 Poutre d'égale résistance, de largeur constante $b$




$$h = h(x_1)$$

EX 3 :




statique : $X_0 = 0$

$$Y_0 - p_2 \frac{\ell}{2} = 0$$

$$M_0 - p_2 \frac{\ell}{2} \frac{3}{4} \ell = 0$$

$$M_0 = \frac{3}{8} p_2 \ell^2$$

1) Diagrammes de $T_2$ et $M_3$.

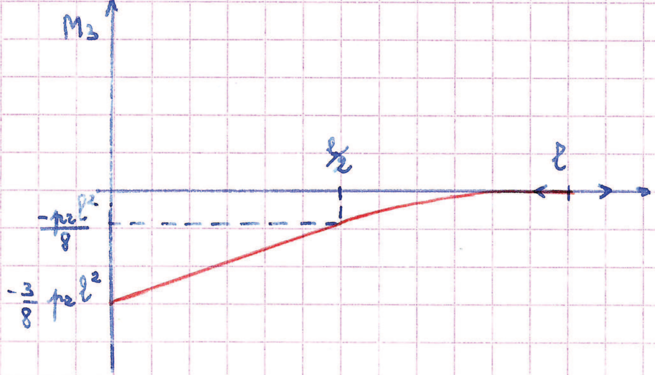
2) Etudier la déformée.

$x \in (0, \frac{\ell}{2})$ $\quad T_2 = -p_2 \frac{\ell}{2}$

$$M_3 = -p_2 \frac{\ell}{2} \times \left(\frac{\ell}{2} + \frac{\ell}{4} - x_1\right) = -p_2 \frac{\ell}{2}\left(\frac{3\ell}{4} - x_1\right)$$

$x \in (\frac{\ell}{2}, \ell)$ $\quad T_2 = -p_2 (\ell - x)$

$$M_3 = -p_2 \frac{(\ell - x_1)^2}{2}$$


Déformée :

$x \in (0, \frac{\ell}{2})$ $\quad \frac{d^2 u_2^{(1)}}{dx_1^2} = \frac{M_3}{E.I_3} = -P_2 \frac{\ell}{2} \left(\frac{3}{4}\ell - x_1\right) \times \frac{1}{EI_3}$

$$\Longrightarrow EI \frac{du_2^{(1)}}{dx_1} = + P_2 \frac{\ell}{4} \left(\frac{3}{4}\ell - x_1\right)^2 + C_1$$

$$\Longrightarrow EI u_2 = - P_2 \frac{\ell}{12} \left(\frac{3}{4}\ell - x_1\right)^3 + C_1 x_1 + C_2$$

$x = 0 \Longrightarrow C_1 = \frac{-9}{64} p_2 \ell^3$

$$C_2 = \frac{9 P_2 \ell^4}{4 \times 64} = \frac{9}{256} p_2 \ell^4 .$$

$x \in (\frac{\ell}{2}, \ell)$

$$EI \frac{d^2 u_2^{(2)}}{dx_1^2} = \frac{-p_2 (\ell - x_1)^2}{2}$$

$$EI \frac{du_2}{dx_1} = \frac{p_2}{6} (\ell - x_1)^3 + C_3$$

$$E.I \; u_2 = -\frac{p_2}{24} (\ell - x_1)^4 + C_3 x_1 + C_4$$

Pour $x = \frac{\ell}{2}$ $\begin{cases} \hat{m} \text{ pente} \\ \hat{m} \text{ flèche} \end{cases} \Rightarrow C_3 = -\frac{1}{8} p_2 \ell^3 - \frac{p_2 \ell^3}{48} = \frac{-7}{48} p_2 \ell^3$

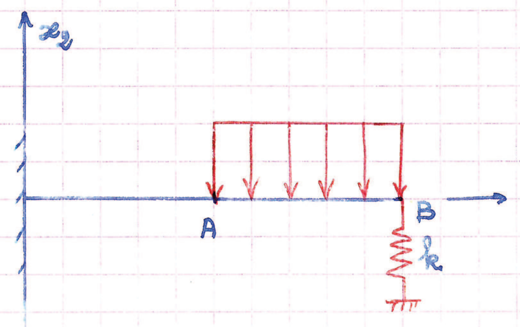
$\hat{m}$ flèche : $\quad -\frac{P_2}{24} \frac{\ell^4}{16} - \frac{7 p_2 \ell^4}{96} + C_4 = -\frac{p_2 \ell}{12} \cdot \frac{\ell^3}{64} - \frac{9 P_2 \ell^4}{2 \cdot 64} + \frac{9}{256} p_2 \ell^4$

$$\Rightarrow C_4 = p_2 \ell^4 \left(\frac{30}{16 \times 48}\right) = \frac{5}{128} p_2 \ell^4 .$$

flèche en B :

$(x = \ell) \Rightarrow u_2(B) = \frac{1}{EI} P_2 \ell^4 \left(\frac{-7}{48} + \frac{5}{128}\right) = -\frac{P_2 \ell^4}{EI} \frac{41}{16 \times 24}$ ;

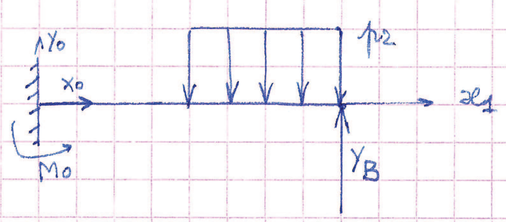
EX 4.




- Réaction en B
- flèche en B

On utilisera le principe de superposition.

Isolons la poutre.

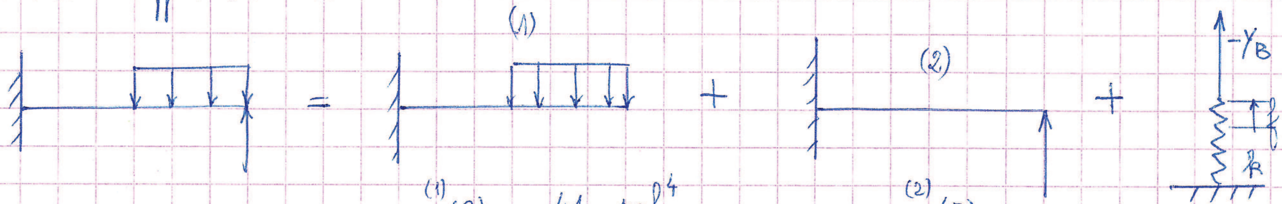



statique :

$$\begin{cases} X_0 = 0 \\ Y_0 + Y_B - p_2 \frac{l}{2} = 0 \\ M_0 + Y_B l - \frac{3}{4} p_2 \frac{l^2}{2} = 0 \end{cases}$$

$Y_0$ , $Y_B$ , $M_0$ $\quad H = 1$

c) équation supplémentaire

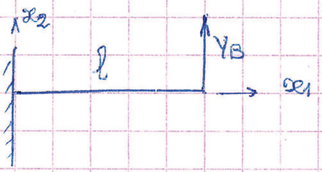



$u_2^{(1)}(B) = \frac{-41}{384} \frac{p_2 l^4}{EI}$ $\qquad u_2^{(2)}(B)$ $\qquad -Y_B = kf$

équation supplémentaire :

$$\begin{cases} f = u_2^{(1)} + u_2^{(2)} = f = \frac{-Y_B}{k} \\ u_2^{(1)} + u_2^{(2)} = \frac{-Y_B}{k} \end{cases}$$

$u_2^{(2)}(B)$ ?




$$M_3 = Y_B (l - x)$$

$$E I_3 \frac{d^2 u_2^{(2)}}{dx_2^2} = Y_B (l - x)$$

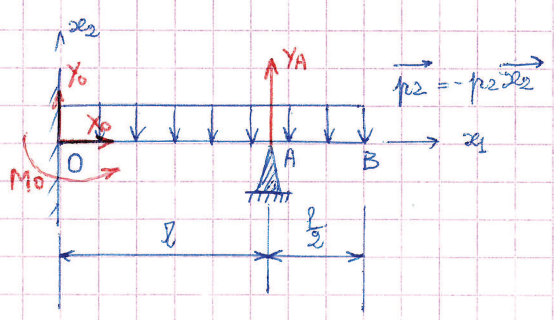
$$E I_3 \frac{du_2}{dx_2} = -\frac{1}{2} Y_B (l - x)^2 + C_1 \rightarrow C_1 = \frac{1}{2} Y_B l^2.$$

$$\rightarrow E I_3 u_2^{(2)} = \frac{1}{6} Y_B (l - x)^3 + \frac{1}{2} Y_B l^2 x + C_2 \rightarrow C_2 = -\frac{1}{6} Y_B l^3.$$

$$\Longrightarrow \frac{-41}{384} \frac{p_2 l^4}{E I_3} + \frac{Y_B l^3}{3 E I_3} = -\frac{Y_B}{k} \cancel{E I_3}$$

$$\rightarrow Y_B = \left[ \frac{41}{128} \frac{p_2 k l^4}{3 E I_3 + k l^3} \right] \qquad f = \frac{-Y_B}{k}$$

<u>EX</u> :




1) Actions inconnues ?

2) diagrammes $T_2$ et $M_3$ !

3) $\omega_3(B)$ ?

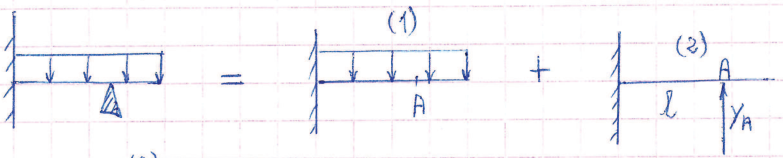
statique :

$$\begin{cases} X_0 = 0 \\ Y_0 + Y_A - p_2 \frac{3l}{2} = 0 \\ M_0 + Y_A l - \frac{9}{8} p_2 l^2 = 0 \end{cases}$$

2 éq 3 inc

$H = 1$

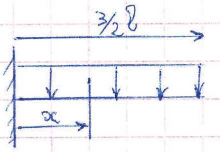
c) équation supplémentaire



$u_2^{(1)}(A) + u_2^{(2)}(A) = 0$

$$u_2^{(2)}(A) = \frac{Y_A l^3}{3EI_3}$$

état 1.



$$M_3 = -p_2 \left(\frac{3}{2}l - x\right)^2 \cdot \frac{1}{2}$$

$$EI_3 \frac{d^2 u_2}{dx_2} = \frac{-p_2}{2}\left(\frac{3}{2}l - x\right)^2$$

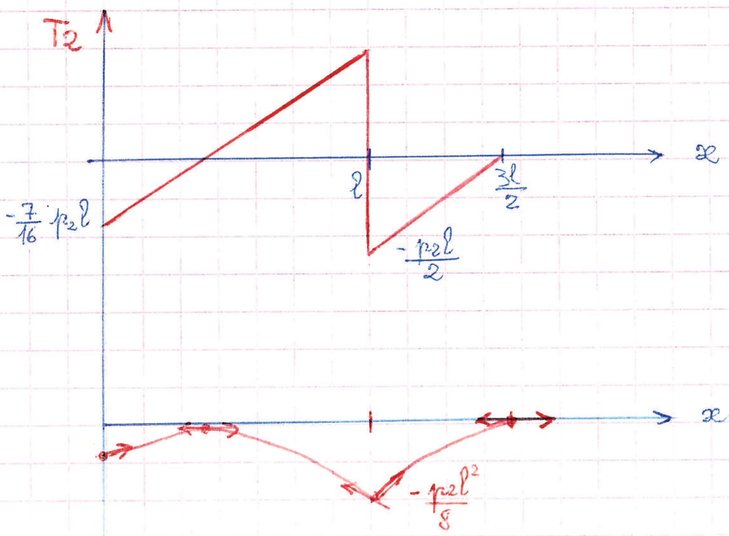
$$\Rightarrow \quad EI_3 \frac{du_2}{dx_2} = \frac{p_2}{6}\left(\frac{3}{2}l - x\right)^3 + C_1 \quad \rightarrow \quad C_1 = \frac{-9}{16} p_2 l^3.$$

$$EI_3 u_2^{(1)} = \frac{-p_2}{24}\left(\frac{3}{2}l - x\right)^4 - \frac{9}{16} p_2 l^3 x + C_2 \quad \rightarrow \quad C_2 = \frac{81}{384} p_2 l^4.$$

$$EI_3 u_2 = \frac{-p_2 l^4}{384} - \frac{9}{16} p_2 l^4 + \left(\frac{384}{81}\right)^{-1} p_2 l^4$$

$$u_2^{(1)}(A) = -\frac{17}{48} \frac{p_2 l^4}{EI_3}$$

$$\frac{Y_A l^3}{3EI_3} - \frac{17}{16}\frac{p_2 l^4}{EI_3} = 0 \quad \rightarrow \quad Y_A = \frac{17}{16} p_2 l.$$

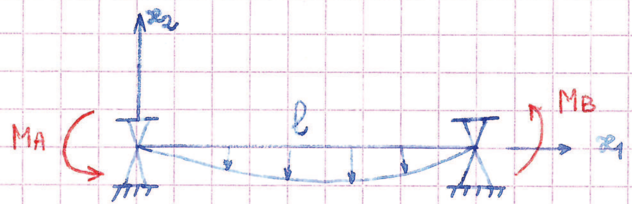


$x \in (0, \ell)$
$$\begin{cases} M_3 = -\dfrac{p_2}{2}\left(\dfrac{3\ell}{2} - x\right)^2 + Y_A(\ell - x) \\ T_2 = -p_2\left(\dfrac{3\ell}{2} - x\right) + Y_A \end{cases}$$

$x \in \left(\ell, \dfrac{3\ell}{2}\right)$
$$\begin{cases} T_2 = -p_2\left(\dfrac{3}{2}\ell - x\right) \\ M_3 = -\dfrac{p_2}{2}\left(\dfrac{3\ell}{2} - x\right)^2 \end{cases}$$

$\dfrac{dM_3}{dx} = -T_2$ tgte horizontale pour $x = \dfrac{3\ell}{2}$.

## EX 1

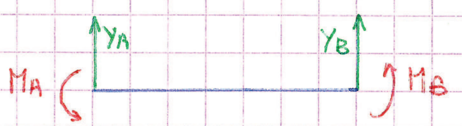



Etablir la matrice :

$$\begin{pmatrix} \omega_A \\ \omega_B \end{pmatrix} = \begin{pmatrix} S \end{pmatrix} \begin{pmatrix} M_A \\ M_B \end{pmatrix}$$ matrice de souplesse

$I_3 = I$

1) Isolons la poutre :




statique
$$\begin{cases} Y_A + Y_B = 0 \\ M_A + M_B + Y_B\ell = 0 \end{cases}$$

$$\Rightarrow Y_B = -\frac{M_A + M_B}{\ell} = -Y_A.$$

### Equation de la déformée :

$$M_3 = M_B + Y_B(\ell - x)$$

$$EI_3 \frac{d^2 u_2}{dx^2} = M_3 = M_B + Y_B(\ell - x)$$

$$EI_3 \frac{du_2}{dx} = M_B x - Y_B \frac{(\ell - x)^2}{2} + C_1$$

$$EI_3 u_2 = M_B \frac{x^2}{2} + Y_B \frac{(\ell - x)^3}{6} + C_1 x + C_2$$

$x = 0$ $\quad C_2 = -\dfrac{Y_B \ell^3}{6}$

$x = \ell$ $\quad \ell C_1 = \dfrac{-M_B \ell^2}{2} + \dfrac{Y_B \ell^3}{6} \Rightarrow C_1 = \dfrac{-M_B \ell}{2} + \dfrac{Y_B \ell^2}{6}$

$$C_1 = -\frac{\ell}{2}\left[M_B + \frac{M_A}{3} + \frac{M_B}{3}\right] = -\frac{\ell}{6}\left(4M_B + M_A\right)$$

$$\frac{dw_2}{dx} = \frac{1}{EI_3}\left[M_B\, x - \frac{Y_B}{2}(\ell - x)^2 - \frac{\ell}{6}(4M_B + M_A)\right]$$

$\underline{x = 0}$ $$\frac{dw_2}{dx} = \omega_A = \frac{1}{EI_3}\left[\frac{M_A + M_B}{2}\,\ell - \frac{\ell}{6}(4M_B + M_A)\right]$$

$$= \frac{\ell}{6EI_3}\left(2M_A - M_B\right)$$

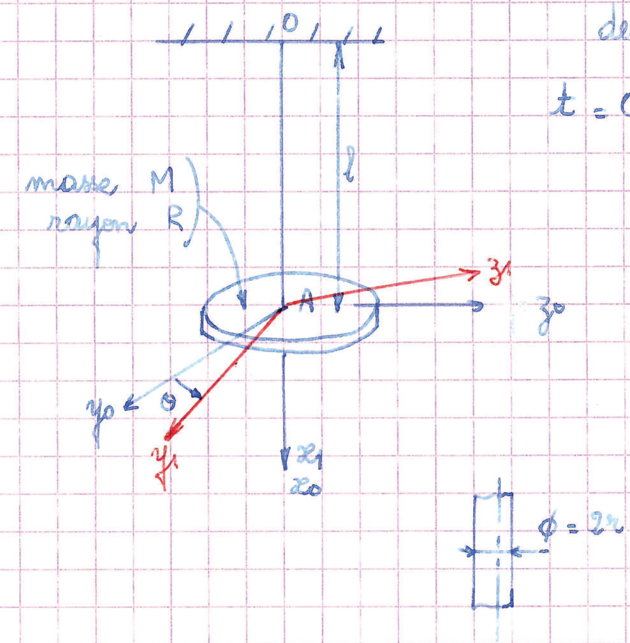
$\underline{x = \ell}$ $$\omega_B = \frac{1}{EI_3}\left[M_B\,\ell - \frac{\ell}{6}(4M_B + M_A)\right] = \frac{\ell}{6EI_3}\left(2M_B - M_A\right)$$

D'où :

$$\begin{pmatrix}\omega_A \\ \omega_B\end{pmatrix} = \underbrace{\frac{\ell}{6EI_3}\begin{vmatrix} 2 & -1 \\ -1 & 2\end{vmatrix}}_{\text{souplesse}} \begin{pmatrix} M_A \\ M_B\end{pmatrix}$$

$$\begin{pmatrix} M_A \\ M_B\end{pmatrix} = [K]\begin{pmatrix}\omega_A \\ \omega_B\end{pmatrix}$$

## EX 2 torsion



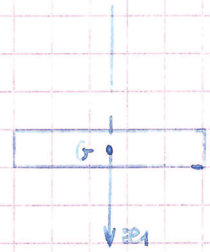

hyp : masse fil négligeable devant celle du disque (D)

$t = 0$ $\begin{cases}\theta = \theta_0 \\ \dot{\theta}_0 = 0\end{cases}$

1) Equation différentielle du mvt

2) en déduire une mesure expérimentale du module G.

1) Isolons le disque (D)



poids : $-Mg\,\vec{x_1}$

$$[\mathcal{T}\ \text{fil}/\text{disque}]_G = \begin{cases} -T\,\vec{x_1} \\ \vec{M}_{F/D} \text{ porté par } \vec{x_1} \end{cases}$$

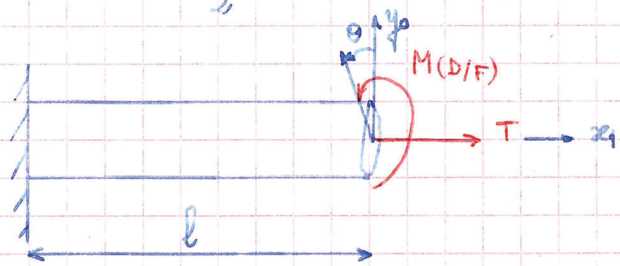
2) Principe fondamental au disque.

th résultante : $\vec{0} = -T\vec{x_1} + Mg\,\vec{x_1} \rightarrow T = Mg.$

th m$^t$ dyn en G : $\vec{0}$ ; $\vec{\sigma}(G) = \begin{vmatrix} \frac{1}{2}MR^2 & & \\ & B & \\ & & C \end{vmatrix} \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}MR^2\dot\theta\,\vec{x_1}$

D'où $\vec{\delta_G} = \frac{1}{2}MR^2\ddot\theta\,\vec{x_1} = \vec{M}$ fil / disque.

$/\vec{x_1}$ $\quad \frac{1}{2}MR^2\ddot\theta = M(F/D).$ (1)




Ici $M_1 = M(D/F)$ et $M_1 \overset{c^{te}}{=} G I_0 \frac{d\omega_1}{dx_1}$

ici $\frac{d\omega_1}{dx_1} = c^{te}$ donc $\frac{d\omega_1}{dx_1} = \frac{\theta}{\ell}$

$\hookrightarrow M_1 = G I_0 \frac{\theta}{\ell}$ avec $I_0 = \frac{\pi R^4}{2}$

(1) $\rightarrow \frac{1}{2}MR^2\ddot\theta = M_{F/D} = -M_1 = -G I_0 \frac{\theta}{\ell}$

$\Rightarrow \frac{1}{2}MR^2\ddot\theta + G I_0 \frac{\theta}{\ell} = 0$

$\Rightarrow \ddot\theta + \underbrace{\frac{2 G I_0}{MR^2 \ell}}_{\Omega^2}\theta = 0$

$\theta = A\cos\Omega t + B\sin\Omega t$

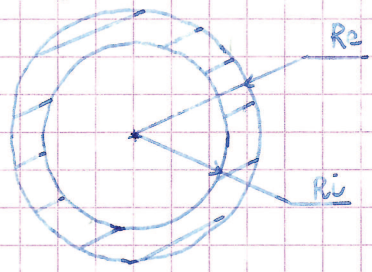
mvt périodique

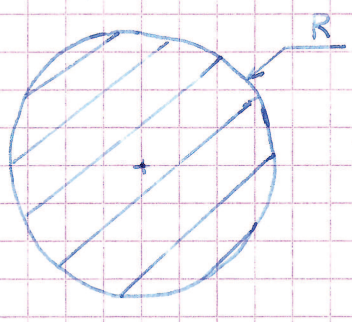
$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{MR^2\ell}{2 G I_0}} \rightarrow T^2 = \frac{4\pi\, MR^2\ell}{G\, r^4}$$

## EX 3

Comparaison entre arbre creux et arbre plein.

$I_0 = \frac{\pi}{2}(Re^4 - Ri^4)$




$I_0 = \frac{\pi R^4}{2}$

$\rho = \frac{Ri}{Re} = 0,6$ 1) Déterminer le rapport $\frac{P_c}{P_p}$

a) A égalité de contrainte maxi.

b) A égalité de déformation

$$\begin{cases} P_c = \pi\,(Re^2 - Ri^2)\,\mu & \text{(longueur unité.)} \\ P_p = \pi R^2 \mu \end{cases}$$

$$\begin{cases} P_c = \pi Re^2 (1-\rho^2)\,\mu \\ P_p = \pi R^2 \mu \end{cases}$$

$\tau = Mt \frac{R}{I_0}$ $\tau$ maxi sur R maxi.

$$\tau^{creux}_{maxi} = \frac{M_t\, Re}{\frac{\pi}{2}(Re^4 - Ri^4)} = \frac{M_t\, Re}{\frac{\pi}{2} Re^4 (1-\rho^4)}$$

$$\tau^{plein}_{maxi} = \frac{M_t}{\frac{\pi}{2} R^3}$$

1er critère : $\tau^c_{maxi} = \tau^p_{maxi}$

$$\frac{1}{R^3} = \frac{1}{Re^3} \cdot \frac{1}{1-\rho^4} \Rightarrow \left(\frac{Re}{R}\right)^3 = \frac{1}{1-\rho^4} \Rightarrow \frac{Re}{R} = \frac{1}{(1-\rho^4)^{1/3}}$$

$$\frac{P_c}{P_p} = \left(\frac{Re}{R}\right)^2 (1-\rho^2) = \frac{1-\rho^2}{(1-\rho^4)^{2/3}} = \frac{(1-\rho^2)^{\frac{1}{3}}}{(1+\rho^2)^{\frac{2}{3}}} = 0,702 \quad (\text{si } \rho = 0,6)$$

A égalité de déformation.

$$\frac{dw_1}{dx} = \frac{M}{G I_0}$$

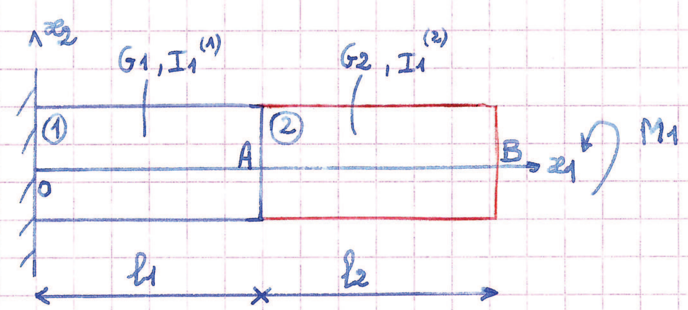
$$\left(\frac{dw_1}{dx}\right)^c = \left(\frac{dw_1}{dx}\right)^p$$

$$\Rightarrow \frac{M_t}{G I_0^p} = \frac{M_t}{G I_0^c} \Rightarrow I_0^p = I_0^c$$

$$\rightarrow R^4 = R_e^4 (1-\rho^4)$$

$$\left(\frac{R_e}{R}\right)^4 = \frac{1}{1-\rho^4} \Rightarrow \frac{P_c}{P_p} = \frac{1-\rho^2}{(1-\rho^4)^{\frac{1}{2}}} = 0{,}86 \quad (\rho = 0{,}6).$$

## EX 1




Déterminer la raideur en torsion

On a : $M_1 = k\, w_1(B)$

<u>sur OA :</u> $$M_1 = G_1 \frac{dw_1^{(1)}}{dx} I_1^{(1)}$$

$$\Rightarrow dw_1^{(1)} = \frac{M_1}{G_1 I_1^{(1)}} dx \rightarrow w_1^{(1)} = \frac{M_1}{G_1 I_1^{(1)}} x + C_1 \qquad \left(\begin{array}{l} w = 0 \\ x = 0 \end{array}\right.$$

$$\rightarrow C_1 = 0$$

D'où $w_1(A) = \dfrac{M_1 l_1}{G_1 I_1^{(1)}}$ ; ou $w_1^{(1)} = \dfrac{M_1 x}{G_1 I_1^{(1)}}$ ;

<u>sur AB</u>

$$M_1 = G_2 \frac{dw_1^{(2)}}{dx} I_1^{(2)}$$

$$\Rightarrow dw_1^{(2)} = \frac{M_1}{G_2 I_1^{(2)}} dx \Rightarrow w_1^{(2)} = \frac{M_1}{G_2 I_1^{(2)}} x + C_2$$

$x = l_1 \rightarrow w_1^{(2)}(A) = w_1^{(1)}(A)$ si bon collage.

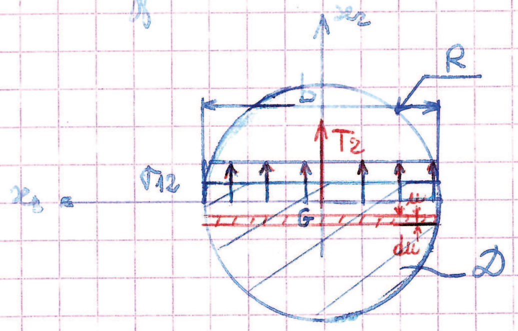
$$\Leftrightarrow \frac{M_1 l_1}{G_1 I_1^{(1)}} = \frac{M_1 l_1}{G_2 I_1^{(2)}} + C_2 \rightarrow C_2 = M_1 l_1 \left(\frac{1}{G_1 I_1^{(1)}} - \frac{1}{G_2 I_1^{(2)}}\right)$$

$$\rightarrow w_1(B) = \frac{M_1}{G_2 I_1^{(2)}} (l_1 + l_2) + M_1 l_1 \left(\frac{1}{G_1 I_1^{(1)}} - \frac{1}{G_2 I_1^{(2)}}\right)$$

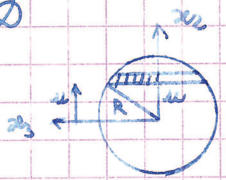
$$\rightarrow w_1(B) = M_1 \left(\frac{l_2}{G_2 I_1^{(2)}} + \frac{l_1}{G_1 I_1^{(1)}}\right) \Rightarrow k = \frac{1}{\dfrac{l_1}{G_1 I_1^{(1)}} + \dfrac{l_2}{G_2 I_1^{(2)}}}$$

## Ex 2

### Effort tranchant




$$\phi_M = -\frac{T_2\, m_3(\mathcal{D})}{I_3}$$
$$\phi_M = \sigma_{12} \cdot b$$
$$\Big\} \quad \sigma_{12} = -\frac{T_2\, m_3(\mathcal{D})}{I_3 \cdot b}$$




$$dm_3 = u \times da = 2u \underbrace{\sqrt{R^2 - u^2}\, du}_{da}$$

(pythagore)

$$m_3(\mathcal{D}) = 2\int_{-R}^{x_2} u\sqrt{R^2 - u^2}\, du$$

changement de variable $\begin{cases} x = u^2 \\ dx = 2u\, du \end{cases}$

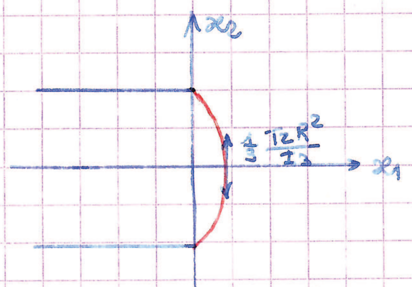
$$\rightarrow \int u\sqrt{R^2-u^2}\,du = \int \sqrt{R^2 - x}\,\frac{dx}{2} = \frac{1}{2}\int \sqrt{R^2-x}\,dx$$

$$= \frac{1}{2}\int (R^2-x)^{\frac{1}{2}}\,dx = -\frac{1}{2}\times\frac{2}{3}(R^2-x)^{\frac{3}{2}} = -\frac{1}{3}(R^2-x)^{\frac{3}{2}}.$$

D'où $$m_3(\mathcal{D}) = 2\left[-\frac{1}{3}(R^2-u^2)^{\frac{3}{2}}\right]_{-R}^{x_2} = -\frac{2}{3}\left[(R^2-x_2^2)^{\frac{3}{2}} - 0\right]$$

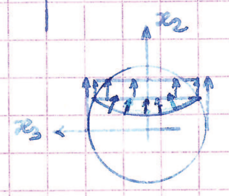
$$\Longrightarrow m_3(\mathcal{D}) = -\frac{2}{3}(R^2-x_2^2)^{\frac{3}{2}}$$

$$\sigma_{12} = +\frac{T_2}{I_3}\,\frac{2}{3}\,\frac{(R^2-x_2^2)^{\frac{3}{2}}}{2(R^2-x_2^2)^{\frac{1}{2}}} = \frac{T_2}{3I_3}(R^2-x_2^2)$$




$$I_3 = \frac{1}{4}SR^2 = \frac{\pi R^4}{4}$$

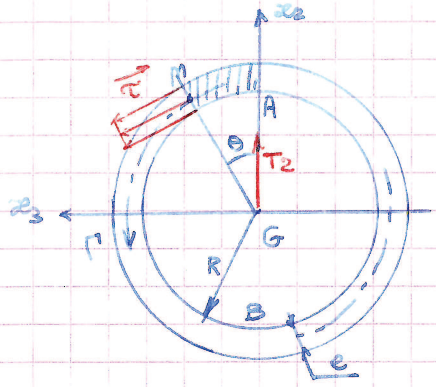
$$\Longrightarrow \sigma_{MAXI} = \frac{1}{3}\,\frac{T_2 R^2}{\frac{1}{4}SR^2} = \frac{4}{3}\,\frac{T_2}{S}$$




## EX 3 :

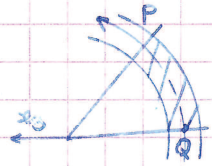
champ des contraintes tgtielles dans la section






$e \ll R$

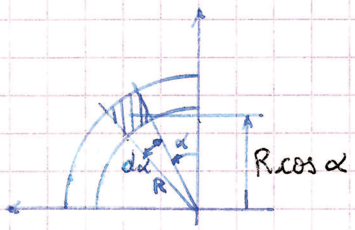
$$\phi_M = e\tau$$




$$\phi_P - \phi_Q = \frac{-T_2 M_3(\mathcal{D})}{I_3}$$

En raison de la symétrie, $\underbrace{\phi_A = \phi_B}_{=0} \longleftrightarrow (T_2 \text{ sur } x_2)$

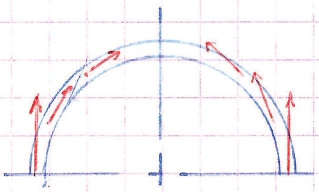
$$\phi_M - \underbrace{\cancel{\phi_A}}_{=0} = \frac{-T_2 M_3(\mathcal{D})}{I_3}$$




$$dm_3 = \overbrace{R\,d\alpha\,e}^{S}\,R\cos\alpha = R^2 e\cos\alpha\,d\alpha$$

$$m_3 = \int_0^\theta eR^2\cos\alpha\,d\alpha = eR^2\sin\theta$$

$$\phi_M = \frac{-T_2\,e\,R^2\sin\theta}{I_3} \longrightarrow \tau_M = \frac{-T_2 R^2\sin\theta}{I_3} < 0 \quad (\text{par rapport à } \Gamma)$$



$$\begin{cases} \tau_{MAXI} \text{ pour } \theta = \frac{\pi}{2} \\ \tau = 0 \quad \theta = 0 \end{cases}$$

$$\tau_{MAXI} = \frac{-T_2 R^2}{I_3};$$

Calcul de $I_3$

$$dI_3 = Re\,d\alpha\,R^2\cos^2\alpha = eR^3\cos^2\alpha\,d\alpha$$

$$I_3 = 2eR^3\int_0^\pi \cos^2\alpha\,d\alpha = 2eR^3\int_0^\pi \frac{\cos 2\alpha + 1}{2}\,d\alpha = eR^3\left[\frac{\sin 2\alpha}{2} + \alpha\right]_0^\pi$$

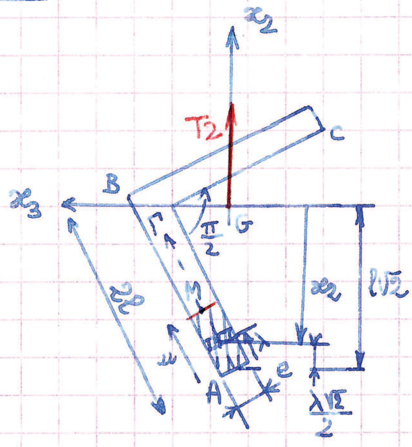
$$I_3 = \pi R^3 e.$$

Ou bien $I_3 = \frac{\pi}{4}\left[(R+e)^4 - R^4\right]$

$S = 2\pi Re \longrightarrow I_3 = S\frac{R^2}{2}$

$$|\tau_{MAXI}| = \frac{T_2 R^2}{I_3} = 2\frac{T_2}{S}$$

## EX 1




$e \ll \ell$

1) Déterminer la répartition des contraintes tangentielles. diagrammes.

2) centre de torsion.

$M \in (AB)$

$$\underbrace{\phi_M - \phi_A}_{\phi_A = 0} = \frac{-T_2\, m_3(\mathcal{D})}{I_3}$$

$$ds = e\,d\lambda \Rightarrow dm_3 = -e\,d\lambda\left(\ell\sqrt{2} - \lambda\frac{\sqrt{2}}{2}\right)$$

(−) car $x_2 < 0$ pour $M \in (AB)$

$$dm_3 = -e\,d\lambda\,\frac{\sqrt{2}}{2}(2\ell - \lambda)$$

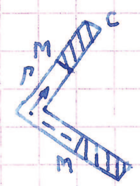
$$\Rightarrow m_3(\mathcal{D}) = -e\frac{\sqrt{2}}{2}\int_0^u (2\ell - \lambda)\,d\lambda = \frac{e\sqrt{2}}{2}\left[\frac{(2\ell-\lambda)^2}{2}\right]_0^u$$

$$m_3(\mathcal{D}) = \frac{e\sqrt{2}}{4}\left[(2\ell-u)^2 - (2\ell)^2\right] = \frac{e\sqrt{2}}{4}(-u)(4\ell - u) = -\frac{e\sqrt{2}}{4}u(4\ell - u)$$

$$\Rightarrow \phi_M = \frac{e\sqrt{2}}{4}\frac{T_2}{I_3}u(4\ell - u)$$

$$\Rightarrow \tau_M = \frac{\sqrt{2}}{4}\frac{T_2}{I_3}u(4\ell - u)$$

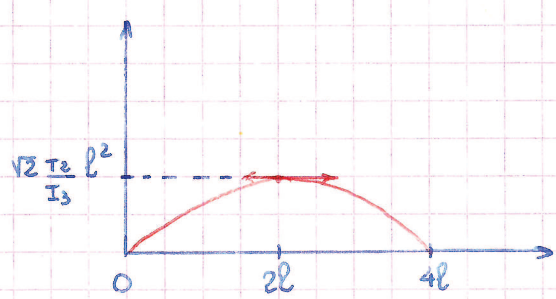
$M \in (BC)$




$$\underset{=0}{\cancel{\phi_C}} - \phi_M = +\frac{T_2}{I_3}\,\underbrace{m_3(\mathcal{D})}_{x_2 > 0\ \ M \in (BC)}$$

D'où la répartition des $\tau$ est symétrique.

$$\tau(2\ell) = \frac{\sqrt{2}}{4}\frac{T_2}{I_3}\,2\ell\,2\ell = \sqrt{2}\,\frac{T_2}{I_3}\ell^2$$




théorie simplifiée

$S = 4\ell e$

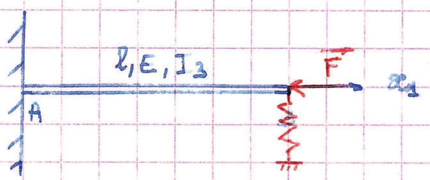
$$\rightarrow \tau_t = \frac{T_2}{4\ell e}$$

On a : $\int_{\Gamma} \overrightarrow{BM} \wedge \tau \, ds = 0$

$(BM) \parallel \tau$

le centre de torsion est en B.

**EX. 2**



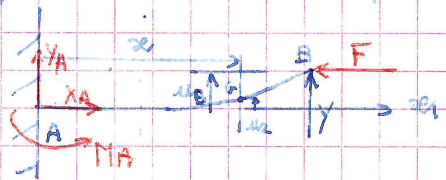

Déterminer l'équation donnant les charges critique.

On a $I_2 \gg I_3$ si direction pour laquelle inertie faible

→ flambage autour de ⊥ cette direction

le plan de flambage est situé dans le plan de la feuille.

1) déformée voisine (algeb > 0)




2) inconnues $X_A, Y_A, M_A, Y$ (action ressort / poutre)

$Y$ algébrique : on le prends > 0 pour simplifier (on cherche pas à comprendre.

3) statique
$$\begin{cases} X_A - F = 0 \\ Y_A + Y = 0 \\ M_A + Yl + F u_B = 0 \end{cases}$$

4) équation déformée

$$E I_3 \frac{d^2 u_2}{dx^2} = M_3 \quad \text{avec} \quad M_3 = Y(l - x) + F(u_B - u_2)$$

$$E I_3 \frac{d^2 u_2}{dx^2} + F u_2 = Y(l - x) + F u_B$$

$$\frac{d^2 u_2}{dx^2} + \frac{F}{EI_3} u_2 = \frac{Y}{EI_3}(l - x) + \frac{F}{EI_3} u_B$$

posons $\omega^2 = \frac{F}{EI_3}$

solution $u_2 = A \cos \omega x + B \sin \omega x$ + sol part 2° membre.

solution particulière avec $2^{nd}$ membre de la forme

$$\begin{cases} u_2^0 = ax + b \\ u_2'^0 = a \\ u_2''^0 = 0 \end{cases}$$

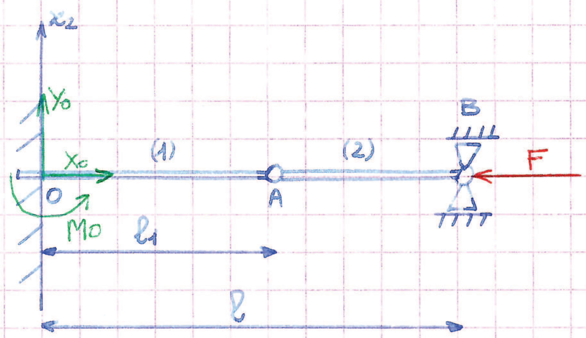
$$\Rightarrow \frac{F}{EI_3}(ax+b) = \frac{Y}{EI_3}(\ell - x) + \frac{F}{EI_3}u_B \Rightarrow \begin{cases} \frac{aF}{EI_3} = \frac{-Y}{EI_3} \rightarrow a = \frac{-Y}{F} \\ \frac{Fb}{EI_3} = \frac{1}{EI_3}(Fu_B + Y\ell) \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{-Y}{F} \\ b = \frac{1}{F}(Fu_B + Y\ell) \end{cases}$$

→ solution générale avec $2^{nd}$ membre

$$u_2 = A\cos\omega x + B\sin\omega x + \frac{Y}{F}(-x+\ell) + u_B$$

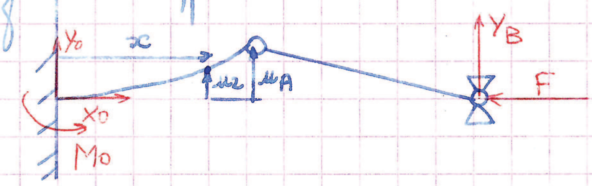
EX 1




(2) indéformable

Déterminer les charges critiques

1) déformée approchée




2) inconnues

3) statique appliquée à (1+2)

$$\begin{cases} X_0 - F = 0 \\ Y_0 + Y_B = 0 \\ M_0 + Y_B\ell = 0 \end{cases} \qquad M_3 = Y_B(\ell - x) - F u_2$$

4) Equation différentielle

$$EI_3 \frac{d^2u_2}{dx^2} = Y_B(\ell - x) - F u_2$$

$$\Rightarrow \quad \frac{d^2u_2}{dx^2} + \frac{F}{EI_3} u_2 = Y_B(\ell - x)\frac{1}{EI_3}$$

l' équation de la déformée de (2) est une dte.

$\left.\begin{array}{l} x = 0 \\ u_2 = 0 \end{array}\right\}$ $u_2 = A \cos \omega x + B \sin \omega x +$ sol particulière avec 2nd membre

sol particulière de la forme

$$u_2^0 = ax + b$$

$\rightarrow$ sol part $\frac{Y_B}{F}(\ell - x)$

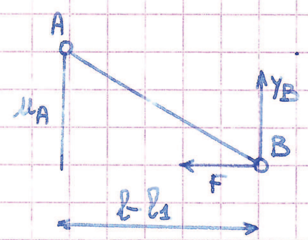
$\rightarrow \quad u_2 = A \cos \omega x + B \sin \omega x + \frac{Y_B}{F}(\ell - x)$

5) C.L

$\left.\begin{array}{l} x = 0 \\ u_2 = 0 \end{array}\right\} \rightarrow \quad A + \frac{Y_B \ell}{F} = 0 \qquad (1)$

$\left.\begin{array}{l} x = 0 \\ \frac{du_2}{dx} = 0 \end{array}\right\} \rightarrow \quad B\omega - \frac{Y_B}{F} = 0 \qquad (2)$

$x = \ell_1 \qquad u_2(\ell_1) = u_A \longrightarrow A \cos \omega \ell_1 + B \sin \omega \ell_1 + \frac{Y_B}{F}(\ell - \ell_1) = u_A \quad (3)$




$$\frac{Y_B}{F} = \frac{u_A}{\ell - \ell_1} \qquad (4)$$

$$\Longrightarrow \quad u_A = \frac{Y_B}{F}(\ell - \ell_1)$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & \frac{\ell}{F} \\ 0 & \omega & -\frac{1}{F} \\ \cos\omega\ell_1 & \sin\omega\ell_1 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ Y_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ système homogène.

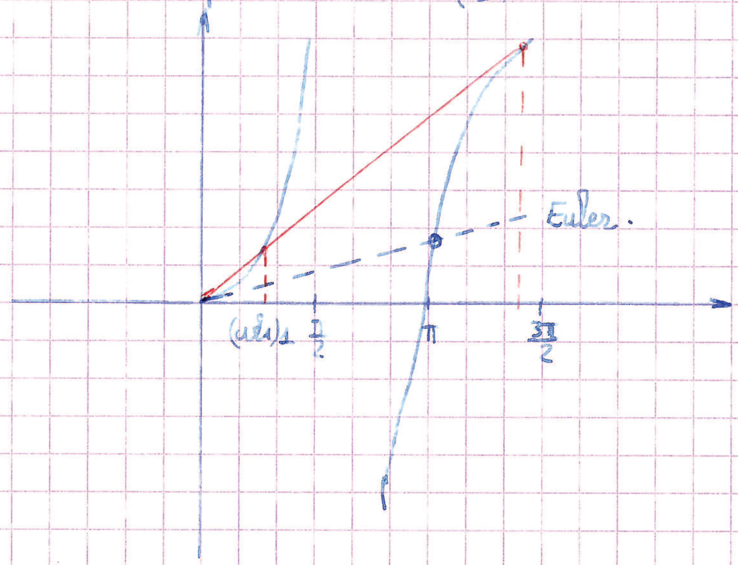
$\det = \frac{1}{F}\sin\omega\ell_1 - \frac{\ell}{F}\omega = 0$ (pour qu'il y ait solutions $\neq 0$ [sinon pas de déformée] il faut $\det = 0$

$$\det = \frac{1}{F}\sin\omega\ell_1 \cdot \cos\omega\ell_1 \cdot \frac{\omega\ell}{F} = 0 \,;$$

$\Rightarrow \quad tg\, \omega l_1 = \omega l$

suite $\infty$ de solution $(\omega_s)$

$tg\, \omega l_1 = \omega l_1 \left(\frac{l}{l_1}\right) \qquad \frac{l}{l_1} > 1$

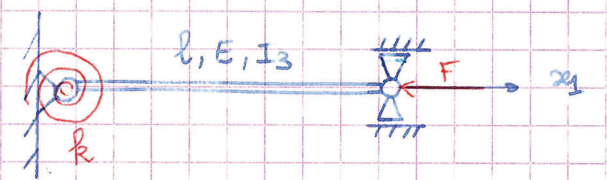



Supposons que $\quad l = 2 l_1$

$\Rightarrow \quad (\omega l_1)_1 \simeq 1{,}2$

$\omega = \frac{1{,}2}{l_1} \quad \Rightarrow \quad F_1 = \frac{1{,}44}{l_1^2} E I_3$

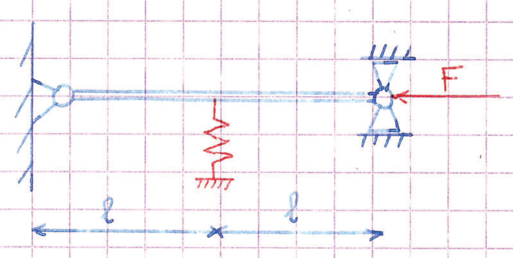
EX 2




charge critique

réponse : $\quad tg\, \omega l = \frac{k \omega l}{k + EI\omega^2 l}$

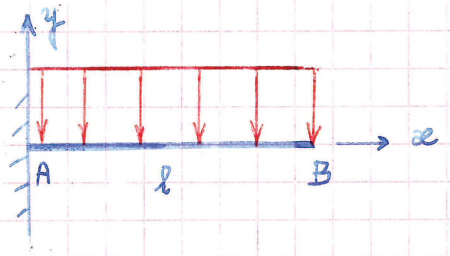
EX 3




charge critique

réponse : $\quad tg\, \omega l = \omega l - \frac{2EI}{kl^3} (\omega l)^3$

**EX 1**




1) Par les formules de Bresse.

a) rotation $\omega_B$.

b) flèche $v_B$.

$$\vec{\omega_B} = \vec{\omega_A} + \int_{\widehat{AB}} \left( \frac{M_3}{EI_3}\vec{x_3} + \frac{M_1}{GI_0}\vec{x_1} + \frac{M_2}{EI_2}\vec{x_2} \right) ds$$

poutre à plan moyen chargée dans son plan

$$M_1 = M_2 = 0$$

en projection $/\vec{z}$ (ou $\vec{x_3}$)

$$\omega_B = \omega_A + \int_{AB} \frac{M_3}{EI} ds$$

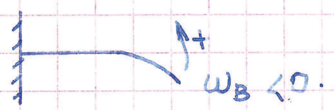
$$v_B = v_A + \omega_A (x_B - x_A) + \int_{\widehat{AB}} \frac{M_3}{EI_3} (x_B - x)\, ds$$

ici $\omega_{3A} = 0$

$$M_3 = -\frac{p(\ell-x)^2}{2}$$

$$\omega_{3B} = \omega_B = \frac{1}{EI}\int_0^\ell -\frac{p(\ell-x)^2}{2} dx = \frac{p}{2EI}\left[\frac{(\ell-x)^3}{3}\right]_0^\ell = \frac{-p\ell^3}{6EI}$$

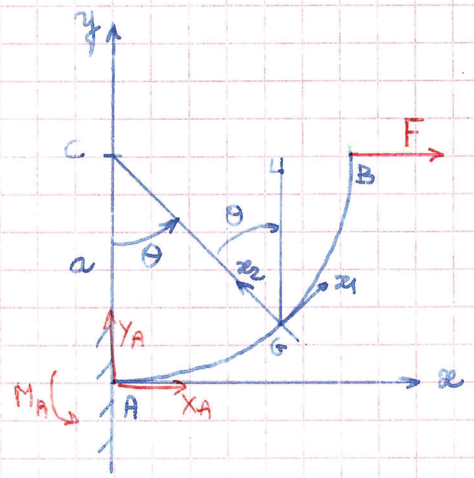
$$\omega_B = \frac{-p\ell^3}{6EI} \qquad (\omega_B < 0)$$




* déplacement (on néglige T2.)

$$v_B = \frac{1}{EI}\int_{\widehat{AB}} -\frac{p}{2}(\ell-x)^2(\ell-x)\, dx = \frac{p}{2EI}\left[\frac{(\ell-x)^4}{4}\right]_0^\ell = \frac{-p\ell^4}{8EI}$$

**EX 2**




section constante, $I_3$, $E$

calculer la rotation de la section B.

calculer le déplacement de B.

$$\begin{cases} \omega_B = \omega_A + \int_{\widehat{AB}} \frac{M_3}{EI_3}\, ds \\ v_B = v_A + \omega_A (x_B - x_A) + \int_{\widehat{AB}} \frac{M_3}{EI_3} (x_B - x)\, ds \\ u_B = u_A - \omega_A (y_B - y_A) - \int_{\widehat{AB}} \frac{M_3}{EI_3} (y_B - y)\, ds \end{cases}$$

statique

$$\begin{cases} X_A + F = 0 \\ Y_A = 0 \\ M_A - aF = 0 \end{cases} \Rightarrow \begin{cases} X_A = -F \\ M_A = aF \end{cases}$$ isostatique.

Rotation $\omega_B = \omega_3(B)$

$M_3 = -aF\cos\theta$ et $ds = a\,d\theta$

$\omega_A = 0$

D'où $\omega_B = \int_0^{\frac{\pi}{2}} \frac{1}{EI} (-aF\cos\theta) \underbrace{a\,d\theta}_{ds} = \frac{-a^2F}{EI} [\sin\theta]_0^{\frac{\pi}{2}}$

$\omega_B = \frac{-a^2F}{EI}$ ;

* $u_B = -\int_{\widehat{AB}} \frac{M_3}{EI} (y_B - y)\, ds = +\frac{1}{EI}\int_0^{\frac{\pi}{2}} aF\cos\theta\,(a-y)\, a\,d\theta$

or $G \left| \begin{array}{l} x = a\sin\theta \\ y = a(1-\cos\theta) \end{array} \right.$

$\Rightarrow u_B = -\frac{1}{EI}\int_0^{\frac{\pi}{2}} (-aF\cos\theta)(a\cos\theta)\, a\,d\theta = \frac{a^3F}{EI}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

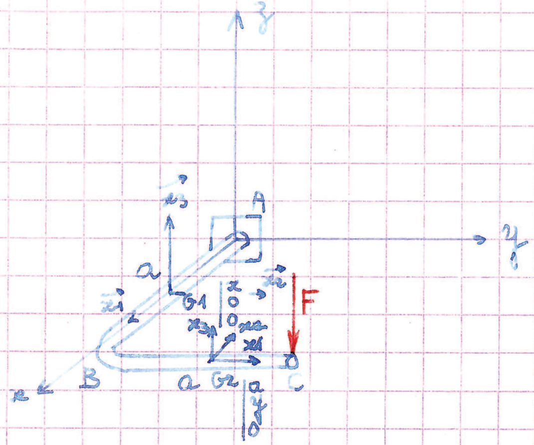
$= \frac{a^3F}{EI}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^3F}{2EI}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} = \frac{\pi a^3 F}{4EI} > 0$ (normal)

* $v_B = \int_{\widehat{AB}} \frac{-aF\cos\theta}{EI_3} (a - x)\, a\,d\theta = \int_{\widehat{AB}} \frac{-a^2F\cos\theta}{EI_3} (a-x)\, d\theta$

$= \frac{-1}{EI}\int_0^{\frac{\pi}{2}} a^2F\cos\theta\,(a - a\sin\theta)\, d\theta = \frac{a^3F}{EI}\int_0^{\frac{\pi}{2}} (\cos\theta\sin\theta - \cos\theta)\, d\theta$

$= \frac{a^3F}{EI}\left[\frac{\sin^2\theta}{2} - \sin\theta\right]_0^{\frac{\pi}{2}} = \frac{-a^3F}{2EI} < 0$ (normal)

EX 1




poutre en équerre encastrée en A

section circulaire uniforme

$\vec{F} = -F\vec{z}$.

1) vecteur rotation $\vec{\omega}$.

2) les composantes du déplacement pt C.

$\vec{u_c}$ $\begin{cases} u_c \\ v_c \\ w_c \end{cases}$

hyp : $\vec{T}$ négligé

1) $x \in (0, a)$ $\qquad \{\mathcal{T}(G_1)\} = \begin{Bmatrix} -F\vec{z} \\ \vec{G_1C} \wedge \vec{F} \end{Bmatrix}_{G_1}$

$$\vec{G_1C} \wedge \vec{F} = \begin{pmatrix} a-x \\ a \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ -F \end{pmatrix} = \begin{pmatrix} -aF \\ (a-x)F \\ 0 \end{pmatrix}$$

Avec la résultante on tire $\begin{cases} N = 0 \\ T_2 = 0 \\ T_3 = -F \end{cases}$ moment $\Rightarrow \begin{cases} M_1 = -aF \\ M_2 = (a-x)F \\ M_3 = 0 \end{cases}$

$$\{\mathcal{T}_2\} = \begin{Bmatrix} -F\vec{z} \\ -(a-x)F\vec{x} \end{Bmatrix}_{G_2} \Rightarrow \begin{cases} N = 0 \\ T_2 = 0 \\ T_3 = -F \end{cases} \text{ et } \begin{cases} M_1 = 0 \\ M_2 = (a-y)F \\ M_3 = 0 \end{cases}$$

Attention aux positions de $\vec{x_1}, \vec{x_2}, \vec{x_3}$ par rapport à $\vec{x}, \vec{y}, \vec{z}$. ($\vec{x_2} = -\vec{x}$ si $G_2$)

Bresse : $$\vec{\omega_C} = \underset{\substack{\| \\ 0}}{\vec{\omega_A}} + \int_{\widehat{AC}} d\vec{\omega} = \int_{\substack{\widehat{AB} \\ 0}}^{a} \left( \frac{-aF}{GI_0}\vec{x} + \frac{(a-x)F}{EI_2}\vec{y} \right) \underbrace{dx}_{= ds} + \int_{\substack{\widehat{BC} \\ 0}}^{a} \frac{-(a-y)F\vec{x}}{EI_2} \underbrace{d}_{ds}$$

car poutre circulaire

$$\Rightarrow \vec{\omega_c} = \frac{-a^2F}{GI_0}\vec{x} + \frac{F}{EI_2}\left(-\frac{(a-x)^2}{2}\right)_0^a \vec{y} + \frac{F}{EI_2}\left(\frac{(a-y)^2}{2}\right)_0^a \vec{x}$$

$$\vec{\omega_c} = \frac{-Fa^2}{2I_2}\left[\frac{1}{G} + \frac{1}{E}\right]\vec{x} + \frac{Fa^2}{2EI_2}\vec{y} \quad \text{ou } I_0 = 2I_2. \text{ (cercle)}$$

2)
$$\vec{v_C} = \vec{v_A} + \vec{\omega_A} \wedge \vec{AC} + \int_{\widehat{AC}} d\vec{\omega} \wedge \vec{GC}$$

Avec $d\vec{\omega} = \left( \frac{M_1}{\ldots}\vec{x_1} + \frac{M_2}{\ldots}\vec{x_2} + \frac{M_3}{\ldots}\vec{x_3} \right) ds$.

$$\rightarrow \quad v_C = \int_{\widehat{AB}} \vec{d\omega} \wedge \vec{G_1C} + \int_{\widehat{BC}} \vec{d\omega} \wedge \vec{G_2C}$$

$$\downarrow$$

$$\left(\frac{-aF}{GI_0}\,\vec{x} + \frac{(a-x)F}{EI_z}\,\vec{y}\right) dx$$

$$\vec{G_1C} \begin{vmatrix} a-x \\ a \\ 0 \end{vmatrix} \qquad \vec{G_2C} \begin{vmatrix} 0 \\ a-y \\ 0 \end{vmatrix}$$

$$\int_{\widehat{AB}} = \int_0^a \left(\frac{-a^2F}{GI_0} - \frac{(a-x)^2F}{EI_z}\right)\vec{z}\, dx + \dots$$

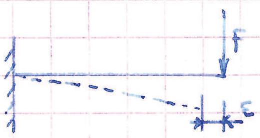
<u>sur BC</u>

$$\dots + \int_0^a -\frac{(a-y)^2F}{EI_z}\,\vec{z}\, dy = \frac{-a^3F}{GI_0}\,\vec{z} + \frac{F}{EI_z}\left[\frac{(a-x)^3}{3}\right]_0^a \vec{z} + \frac{F}{EI_z}\left[\frac{(a-y)^3}{3}\right]_0^a \vec{z}$$

$$= \frac{a^3F}{I_z}\left[-\frac{1}{2G} - \frac{2}{3E}\right]\vec{z} = \frac{-a^3F}{I_z}\left(\frac{1}{2G} + \frac{2}{3E}\right)\vec{z}$$

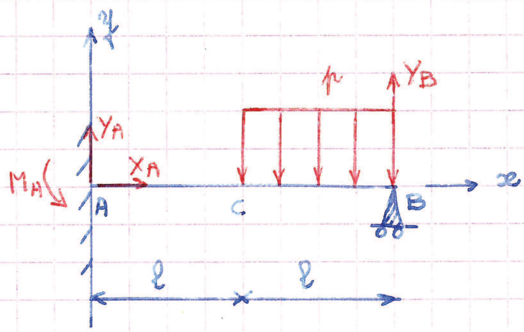
$$\Longrightarrow \begin{cases} u_C = 0 \\ v_C = 0 \\ w_C = \dfrac{-Fa^3}{I_z}\left(\dfrac{1}{2G} + \dfrac{2}{3E}\right) \end{cases}$$

<u>Rque</u> :

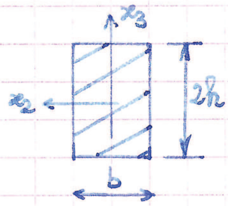



$M_z = -(\ell - x)\,F$

## <u>EX 2</u>




1) calculer les actions inconnues
2) Diagramme des $M^t$ fléchissant
3) contrainte maximale en flexion




statique
$$\begin{cases} X_A = 0 \\ Y_A + Y_B - p\ell = 0 \\ M_A + 2\ell Y_B - \frac{3}{2}\ell^2 p = 0 \end{cases}$$

3 équations 4 inconnues

$\rightarrow$ H = 1

On remarque que $v_B = v_A = \omega_A = 0$

$$\underset{0}{\underset{\parallel}{v_B}} = \underset{0}{\underset{\parallel}{v_A}} + \underset{0}{\underset{\parallel}{\omega_A}}\,(x_B - x_A) + \int_{\widehat{AB}} \frac{M_3}{EI_3}\,(x_B - x)\, dx$$

$x \in (0, \ell) \qquad M_3 = -M_A + Y_A x$

$x \in (\ell, 2\ell) \qquad M_3 = Y_B(2\ell - x) - \frac{p}{2}(2\ell - x)^2$

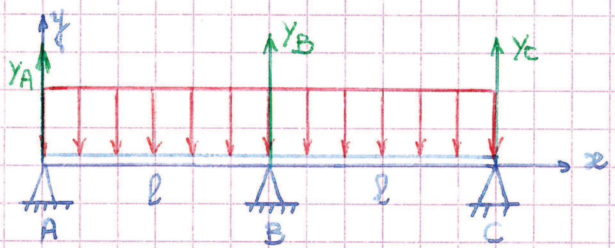
$$\rightarrow \quad 0 = \int_0^\ell (-M_A + Y_A x)(2\ell - x)\,dx + \int_\ell^{2\ell} \left[ Y_B (2\ell - x)^2 - \frac{p}{2}(2\ell - x)^3 \right] dx$$

$$\Rightarrow \quad M_A \left[ \frac{(2\ell - x)^2}{2} \right]_0^\ell + Y_A\left(\ell^3 - \frac{\ell^3}{3}\right) + \left[ -Y_B \frac{(2\ell - x)^3}{3} + \frac{p}{2}\frac{(2\ell - x)^4}{4} \right]_\ell^{2\ell} = 0$$

$$\frac{M_A}{2}(\ell^2 - 4\ell^2) + \frac{2}{3} Y_A \ell^3 + \frac{Y_B}{3}\ell^3 - \frac{p\ell^4}{8} = 0$$

$$\Leftrightarrow \quad -\frac{3}{2} M_A \ell^2 + \frac{2}{3} Y_A \ell^3 + \frac{1}{3} Y_B \ell^3 - \frac{p\ell^4}{8} = 0 \quad \text{(équation suplémentaire)}$$

**EX 1**




1) Determinez les actions inconnues.

2) Diagrammes de $M_3$ et $T_2$.

Statique : $\begin{cases} Y_A + Y_B + Y_C - 2\ell P = 0 \\ Y_B \cdot \ell + 2\ell Y_C - 2\ell P \ell = 0 \end{cases}$ 2 équations, 3 inconnues.

la symétrie donne $Y_A = Y_C$ (elle n'amène rien de plus.)

Donc $H = 1$.

Equation suplémentaire

On a $\omega_B = 0$ (dans l'axe de symétrie) et $v_A = v_B = v_C = 0$

entre B et C

$$\underset{0}{v_C} = \underset{0}{v_B} + \omega_B \underset{0}{(x_C - x_B)} + \int_{\widehat{BC}} \frac{M_3}{EI}(x_C - x)\underbrace{dx}_{= ds}$$

$$0 = \int_{\ell\ (B\to C)}^{2\ell} M_3 (2\ell - x)\,dx \qquad \text{Avec} \quad M_3 = Y_C(2\ell - x) - \frac{P(2\ell - x)^2}{2}.$$

$$\Rightarrow \quad 0 = \int_\ell^{2\ell} \left( Y_C(2\ell - x) - \frac{P(2\ell - x)^2}{2} \right)(2\ell - x)\,dx = \int_\ell^{2\ell} \left[ Y_C (2\ell - x)^2 - \frac{P(2\ell - x)^3}{2} \right] dx$$

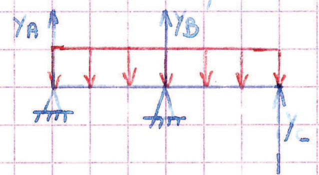
$$0 = \left[ Y_C \frac{(2\ell - x)^3}{3} - \frac{P(2\ell - x)^4}{8} \right]_\ell^{2\ell} = -Y_C \frac{\ell^3}{3} + \frac{P\ell^4}{8} = 0 \rightarrow Y_C = \frac{P\ell^4}{8} \times \frac{3}{\ell^3} = \frac{3P\ell}{8}$$

$Y_A = Y_C = \frac{3P\ell}{8}$ et $Y_B = \frac{5}{4}P\ell$

2° méthode

Ménabréa :

1) Adopter un système isostatique équivalent




force arbitraire remplaçant l'appui : $Y_C$ (inc hyperstatique) statique inchangée.

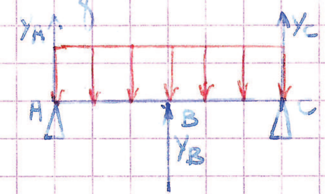
Equation supplémentaire

$$\frac{\partial W}{\partial Y_C} = 0$$

$$W = \frac{1}{2}\int_{ABC} \frac{M_3^2}{EI}\, ds$$

$x \in (0, \ell)$ $\quad M_3 = -\left[-Y_A x + \frac{Px^2}{2}\right] = Y_A x - \frac{Px^2}{2}$

solution plus judicieuse :




$W(ABC) = 2\,W(BC)$

potentiel élastique.

$$M_3 = Y_A x - \frac{Px^2}{2}$$

statique :

$$2Y_A + Y_B - 2P\ell = 0 \Rightarrow Y_A = \frac{2P\ell - Y_B}{2} = P\ell - \frac{Y_B}{2}$$

$$\Rightarrow M_3 = \left(P\ell - \frac{Y_B}{2}\right)x - \frac{Px^2}{2}$$

$$W = 2\int_{AB} \frac{M_3^2}{2EI}\, dx \quad \text{or} \quad \frac{\partial W}{\partial Y_B} = \frac{\partial}{\partial Y_B}\int_{AB} \frac{M_3^2}{EI}\, dx$$

$$= \int_{AB} \frac{\partial}{\partial Y_B}\frac{M_3^2}{EI}\, dx = \frac{1}{EI}\int_{AB} 2M_3 \frac{\partial M_3}{\partial Y_B}\, dx = \frac{\partial W}{\partial Y_B} = 0 \quad \text{où} \quad \frac{\partial M_3}{\partial Y_B} = -\frac{x}{2}.$$

$$\Rightarrow \int_0^\ell \left\{\left[P\ell - \frac{Y_B}{2}\right]x - \frac{Px^2}{2}\right\} x\, dx = 0 \Leftrightarrow \left[\left(P\ell - \frac{Y_B}{2}\right)\frac{x^3}{3} - \frac{Px^4}{8}\right]_0^\ell = 0$$

$$\Rightarrow \left(P\ell - \frac{Y_B}{2}\right)\frac{\ell^3}{3} - \frac{P\ell^4}{8} = 0 \Rightarrow Y_B = \frac{5}{4}P\ell.$$

le moment fléchissant doit être exprimé en fonction de l'inconnue hyperstatique.

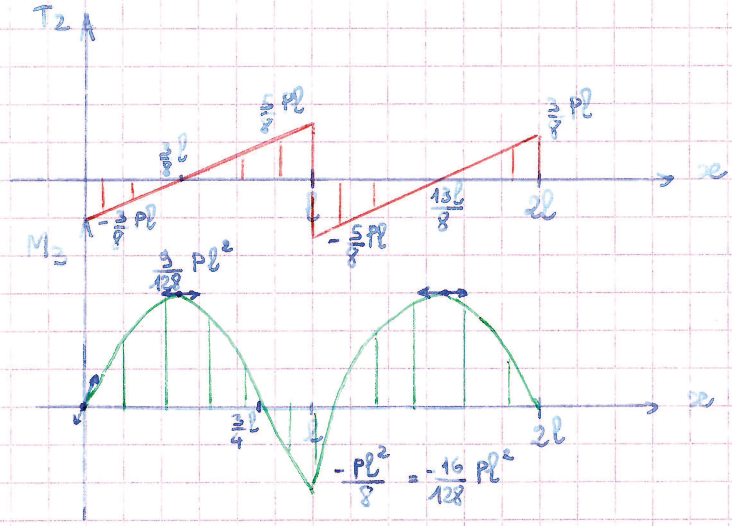
2) Diagrammes : $0 < x < \ell$ $\quad M_3 = Y_A x - \frac{Px^2}{2} = \frac{3}{8}P\ell x - \frac{Px^2}{2}$

$$M_3 = P\frac{x}{2}\left[\frac{3}{4}\ell - x\right]$$

$$T_2 = Px - \frac{3}{8}P\ell = -Y_A + Px = P\left(x - \frac{3}{8}\ell\right)$$

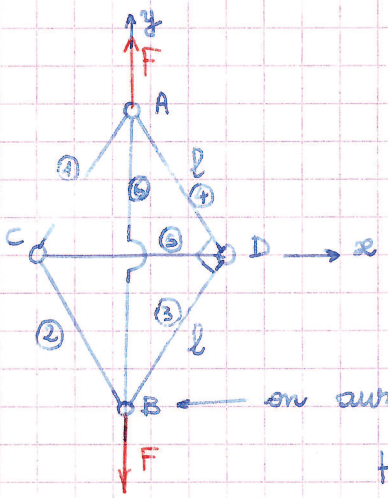
$$\begin{cases} T_2(0) = -\frac{3}{8}P\ell \\ T_2(\ell) = \frac{5}{8}P\ell \end{cases}$$

$$\frac{dM_3}{dx} = -T_2$$




$$M_3\left(\frac{3\ell}{8}\right) = \frac{9}{128}P\ell^2 \quad \text{et} \quad M_3(\ell) = -\frac{P\ell^2}{8}$$

EX 1




1°) efforts dans les barres.

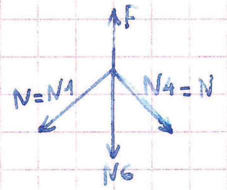
2) déplacement du point A.

Ici $H_e = 0$

← on aurait pu mettre une pivot

$$H_i = b - (2n - 3) = 1 \qquad (b = 6 \quad n = 4)$$

noeud A

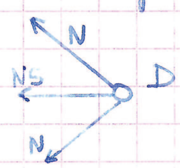



proj /y : $F - N_6 - N\sqrt{2} = 0$ (1)

barres tendues (elles tirent sur le noeud)

pour des raisons de symétrie $N_1 = N_2 = N_3 = N_4 = N$ (m̂ matière partout.)

noeud D

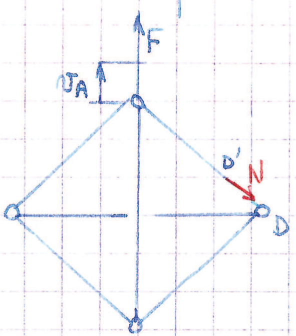



$$-N_5 - N\frac{\sqrt{2}}{2} - N\frac{\sqrt{2}}{2} = 0$$

$$N_5 = -N\sqrt{2} \quad (2)$$

D'où
$$\begin{cases} F - N_6 - N\sqrt{2} = 0 \\ N_5 + N\sqrt{2} = 0 \end{cases}$$

rendons isostatique → supprimer un axe.




système isostatique équivalent : on libère D de (4) (dplt D et D' = 0.)

$\frac{\partial W}{\partial N} = 0 \qquad W = W_1 + W_2 + W_3 + W_4 + W_5 + W_6.$

$$W = \frac{4}{2}\frac{N^2 l}{ES} + \frac{N_5^2 l\sqrt{2}}{2ES} + \frac{N_6^2 l\sqrt{2}}{2ES}$$

$$W = \frac{l}{ES}\left(2N^2 + \frac{\sqrt{2}}{2}2N^2 + \frac{\sqrt{2}}{2}(F - N\sqrt{2})^2\right)$$

$$\frac{\partial W}{\partial N} = 0 \implies 4N + 2\sqrt{2}N + \sqrt{2}(F - N\sqrt{2})(-\sqrt{2}) = 0$$

$$\implies N(2+\sqrt{2}+\sqrt{2}) = F \implies N = \frac{F}{2(1+\sqrt{2})} > 0 \implies \begin{cases} \text{barres } 1,2,3,4 \\ \text{tendues} \end{cases}$$

$$\begin{cases} N_5 = \frac{-F\sqrt{2}}{2(1+\sqrt{2})} < 0 \\ \text{barre 5 comprimée} \end{cases} \qquad \begin{cases} N_6 = \frac{F(2+\sqrt{2})}{2(1+\sqrt{2})} > 0 \\ \text{barre 6 tendue.} \end{cases}$$

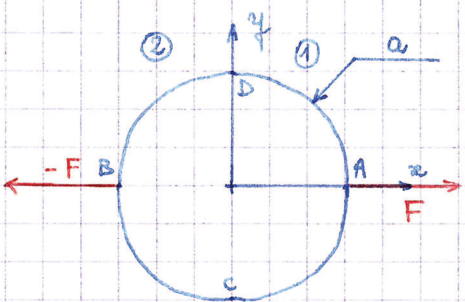
* $v_A = \frac{\partial W}{\partial F}$ (Castigliano.)

* $N_6 = \frac{(2+\sqrt{2})F}{2(1+\sqrt{2})}$

hooke → $\varepsilon = \frac{1}{E}\sigma \rightarrow \frac{v_A}{l\sqrt{2}} = \frac{1}{E}\frac{(2+\sqrt{2})F}{2(1+\sqrt{2})S}$ on l'applique sur la barre 6.

$$\implies v_A = \frac{\sqrt{2}(2+\sqrt{2})}{2(1+\sqrt{2})}\frac{Fl}{ES} = \frac{Fl}{ES}$$

## EX 2

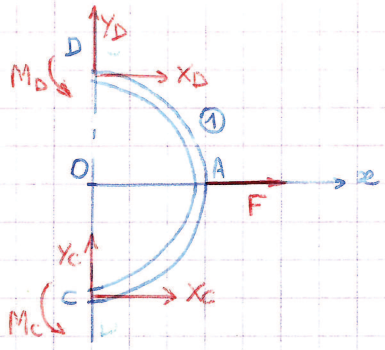



1) degré d'hyperstaticité

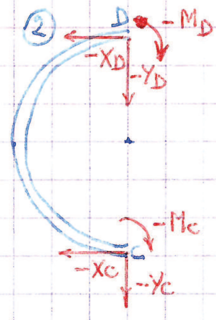
2) déplacement de A.

On a He = 0

faisons une coupure suivant CD :




symétrie :



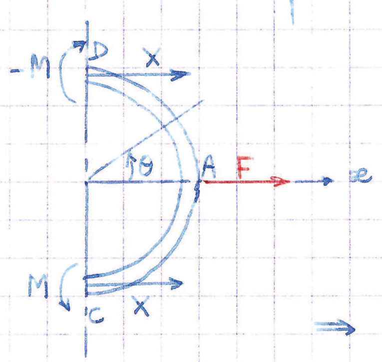

On voit que les 2 parties sont symétriques : par rapport à $Oy$

$Y_D$ et $-Y_D$ ne sont pas symétriques. $\Rightarrow$ $Y_D = 0$

l'effort tranchant est tjs dans la section de symétrie.

symétrie / $Ox$ $\Longrightarrow$ $\begin{cases} X_C = X_D = X \\ M_C = -M_D = M \end{cases}$

Donc en conclusion on adopte le schéma :




statique :

$$\left.\begin{aligned} 2X + F &= 0 \\ mt/_C : \quad M - aF - 2aX - M &= 0 \end{aligned}\right\}$$

$\Rightarrow$ $-a(F + 2X) = 0$ $\Rightarrow$ $X = \frac{-F}{2}$

On ne peut pas déterminer M par la statique.

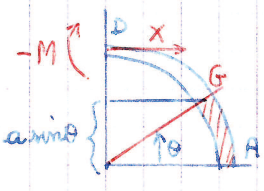
$\Longrightarrow$ $Hi = 1$

Equation supplémentaire :

Bresse : $\begin{cases} v_A = 0 \\ \omega_A = 0 \end{cases}$ $\rightarrow$ (la section se déplace mais ne tourne pas)

$\begin{cases} u_D = 0 \\ \omega_D = 0 \end{cases}$ on tire symétriquement.

faisons $\underset{0}{\underset{\shortparallel}{\omega_D}} = \underset{0}{\underset{\shortparallel}{\omega_A}} + \int_{AD} \frac{M_3}{EI} ds$ $\Rightarrow$ $\int_{AD} \frac{M_3}{EI} ds = 0$




$M_3 = -M - a(1 - \sin\theta) X = -M + \frac{aF}{2}(1 - \sin\theta)$.

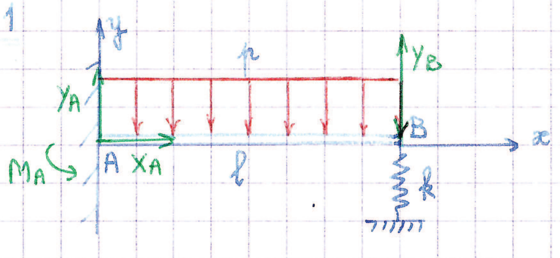
$ds = a d\theta$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[-M + \frac{af}{2}(1-\sin\theta)\right] a\,d\theta = -M\frac{\pi}{2} + \frac{af}{2}\left[\theta + \cos\theta\right]_0^{\frac{\pi}{2}} = 0$$

$$\Rightarrow -M\frac{\pi}{2} + \frac{af}{2}\left[\frac{\pi}{2} - 1\right] = 0$$

$$\Rightarrow M = af\left(\frac{1}{2} - \frac{1}{\pi}\right) > 0$$

Ex 1




1) actions inconnues.

2) Ménabréa.

3) 3 moments.

statique :

$Y_B$ : action ressort sur la poutre.

$$\begin{cases} X_A = 0 \\ Y_A + Y_B - p\ell = 0 \\ M_A + Y_B \ell - \frac{p\ell^2}{2} = 0 \end{cases} \Rightarrow H = 1$$

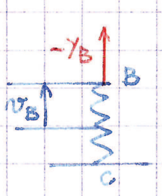
Equation supplémentaire

1) Bresse $\quad v_B = \underset{0}{\underset{\|}{v_A}} + \underset{0}{\underset{\|}{\omega_A}}(x_B - x_A) + \int_{AB} \frac{M_3}{EI}(x_B - x)\,dx$

$$M_3 = (\ell - x)Y_B - p\frac{(\ell-x)^2}{2} \Rightarrow v_B = \frac{1}{EI}\int_0^\ell \left[Y_B(\ell-x)^2 - \frac{p}{2}(\ell-x)^3\right]dx$$

$$\Rightarrow v_B = \frac{1}{EI}\left\{-\frac{Y_B}{3}(\ell-x)^3 + \frac{p}{8}(\ell-x)^4\right\}_0^\ell \Rightarrow EI\,v_B = \frac{Y_B}{3}\ell^3 - \frac{p\ell^4}{8}$$

Ressort

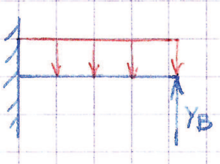



$$\Rightarrow -Y_B = k\,v_B$$

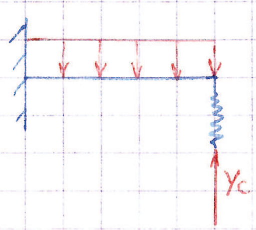
$$\Rightarrow EI\,v_B = \frac{-k\,v_B\,\ell^3}{3} - \frac{p\ell^4}{8}$$

$$\Rightarrow \quad Y_B = \frac{3}{8}\frac{pk\ell^4}{(3EI + k\ell^3)}$$

2) Ménabréa :



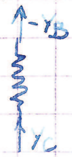

ou




$$\frac{\partial W(\text{poutre} + \text{ressort})}{\partial Y_C} = 0$$

$$\frac{\partial W_{\text{poutre}}}{\partial Y_B} = v_B$$

$$\begin{cases} W_{poutre} = \displaystyle\int_{AB} \frac{M_3^2}{2EI}\,dx \\ W_{ressort} = \dfrac{1}{2} k\, v_B^2 \end{cases}$$




(poutre/R)

(bati/R)

$\Rightarrow \quad Y_C - Y_B = 0$

$Y_B = Y_C.$

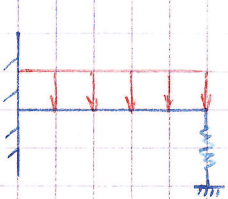
$$W_{ressort} = \frac{1}{2}\frac{Y_B^2}{k} = \frac{1}{2}\frac{Y_C^2}{k}$$

ici $\dfrac{\partial W}{\partial Y_C} = \dfrac{\partial}{\partial Y_C}\displaystyle\int_{AB}\frac{M_3^2}{2EI}\,dx + \frac{\partial}{\partial Y_C}\left(\frac{1}{2}\frac{Y_C^2}{k}\right)$

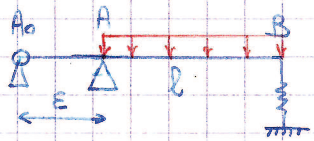
$$\frac{\partial W}{\partial Y_C} = \frac{1}{EI}\int_{AB} M_3 \frac{\partial M_3}{\partial Y_C}\,dx + \frac{Y_C}{k} = \frac{1}{EI}\int_0^{\ell}\left[(\ell-x)Y_C - \frac{p}{2}(\ell-x)^2\right](\ell-x)\,dx + \frac{Y_C}{k} = 0$$

$$= \frac{1}{EI}\left\{-Y_C\frac{(\ell-x)^3}{3} + \frac{p}{8}(\ell-x)^4\right\}_0^{\ell} + \frac{Y_C}{k} = 0 \implies Y_C = \frac{3}{8}\frac{pk\ell^4}{(3EI + k\ell^3)}$$

3) théorème des 3 moments

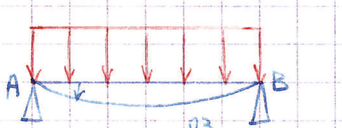


$\Longleftrightarrow$




$$\underset{0}{M_0}\,\varepsilon + 2M_A(\varepsilon+\ell) + \underset{0}{M_B}\,\ell = 6EI(\theta_A^+ - \underset{0}{\theta_A^-}) + 6EI\left[\frac{v_B - \overset{0}{v_A}}{\ell} + \frac{\overset{0}{v_0}}{\varepsilon} \ldots\right.$$

formulaire :




$\theta_A^+ = -\dfrac{p\ell^3}{24EI}\ (<0)$

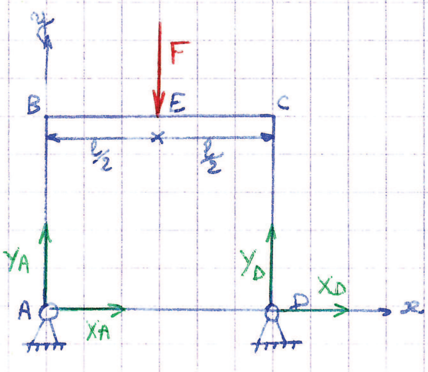
$$\Longrightarrow \quad 2M_A\ell = -\frac{p\ell^3}{4} + \frac{6EI}{\ell}v_B$$

$$\text{or} \quad M_A = Y_B\ell - \frac{p\ell^2}{2}$$

D'où $\quad 2\left(Y_B\ell - \dfrac{p\ell^2}{2}\right)\ell = -\dfrac{p\ell^3}{4} - \dfrac{6EI}{\ell}\dfrac{Y_B}{k}$

$$\Longrightarrow \quad Y_B = \frac{3kp\ell^4}{8(k\ell^3 + 3EI)}$$

## EX 1




1) Actions inconnues.

2) déplacement du point E.

statique
$$\begin{cases} X_A + X_D = 0 \\ Y_A + Y_D - F = 0 \\ -\frac{F\ell}{2} + Y_D \ell = 0 \end{cases} \qquad \begin{cases} X_A = -X_D \\ Y_A = Y_D = \frac{F}{2} \end{cases} \Rightarrow H = 1$$

soit $X_A = X$ inconnue hyperstatique.

$\frac{\partial W}{\partial X} = 0$ (pt A dplt nul)

poutre symétrique (chargement et géométrie)

$$W = 2 W_{AE}$$

$$W_{AE} = \frac{1}{2}\int_{AE} \frac{M_3^2}{EI}\, ds$$

sur AB :

$$m_3 = -[X_A y] = -Xy \qquad (ds = dy)$$

sur le tronçon BE
$$m_3 = -[Xh - Y_A x] \qquad (ds = dx)$$
$$M_3 = -Xh + \frac{F}{2}x$$

$$\frac{\partial W}{\partial X} = \frac{2}{EI}\Big[\int_A^B \underset{mf}{(-Xy)}\ \underset{\frac{\partial mf}{\partial X}}{(-y)}\, dy + \int_B^E \left(\frac{F}{2}x - Xh\right)(-h)\, dx = 0 \qquad (\text{dplt pt } A = 0.)$$

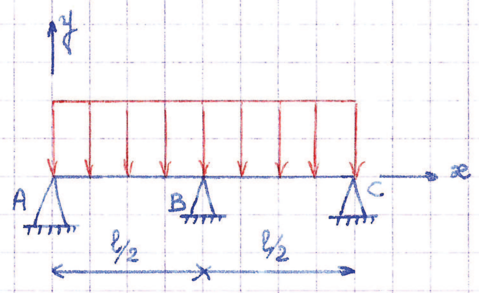
$$\Rightarrow \left[X\frac{y^3}{3}\right]_0^h - \frac{Fh}{2}\int_0^{\frac{\ell}{2}} x\, dx + Xh^2\int_0^{\frac{\ell}{2}} dx = 0$$

$$\Rightarrow Xh\left(\frac{h}{3} + \frac{\ell}{2}\right) = \frac{F\ell^2}{16} \Rightarrow X = \frac{3F\ell^2}{8(2h+3\ell)h};$$

$\frac{\partial W}{\partial F} = v_E$ on doit trouver $v_E > 0$ (suivant la force.)

par Bresse $v_E < 0$ (suivant les axes.)

## EX2




3 moments

$$\underset{0}{\underset{\shortparallel}{M_A}} \frac{l}{2} + 2M_B l + \underset{0}{\underset{\shortparallel}{M_C}} \frac{l}{2} = 6EI(\theta_B^+ - \theta_B^-) \quad \text{or} \quad \begin{cases} \theta_B^+ = \dfrac{-pl^3}{192EI} < 0 \\ \theta_B^- = \dfrac{pl^3}{192EI} > 0 \end{cases}$$

$$2M_B l = 6EI\left(\frac{-pl^3}{192EI} - \frac{pl^3}{192EI}\right)$$

$$\Rightarrow \quad 2M_B l = 6EI\left(\frac{-pl^3}{96EI}\right) \quad \Rightarrow \quad M_B = \frac{-3EI\,pl^2}{96EI} = \frac{-pl^2}{32}$$

or $M_B = Y_C \frac{l}{2} - \frac{pl^2}{8} = \frac{-pl^2}{32}$

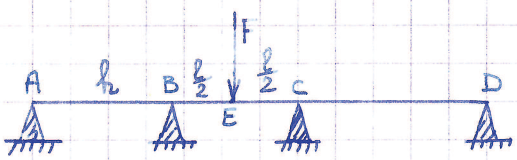
$$\Longrightarrow \quad \frac{1}{2} Y_C = \frac{-pl}{32} + \frac{pl}{8} = \frac{3pl}{32} \quad \Rightarrow \quad Y_C = \frac{6pl}{32} = \frac{3}{16} pl.$$

$\Rightarrow$ avec statique les autres inconnues.

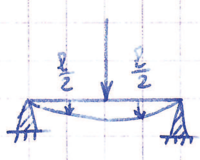
Revenons à l'exo 1

3 moments, les pts B et C ne bouge pas si on néglige T et N.

$H = 1$ statique $\begin{cases} X_A + X_D = 0 \\ Y_A + Y_D - F = 0 \\ -\dfrac{Fl}{2} + Y_D l = 0 \end{cases}$ $\Rightarrow$ symétrie $\begin{cases} Y_A = Y_D \\ X_A = -X_D \end{cases}$




$$\underset{0}{\underset{\shortparallel}{M_A}} h + 2M_B(h+l) + M_C l = 6EI(\theta_B^+ - \underset{0 \text{ (pas chargé)}}{\underset{\shortparallel}{\theta_B^-}}) + 0$$




$\theta_B^+ = \dfrac{-Fl^2}{16EI}$ ici $M_B = M_C$

$$\Rightarrow \quad 2M_B(h+l) + M_B l = 6EI\left(\frac{-Fl^2}{16EI}\right)$$

$$\Rightarrow \quad M_B = -\frac{3}{8} \frac{Fl^2}{(2h+3l)}. \quad \text{donc} \quad X_A = \frac{-M_B}{h}$$

$$\Rightarrow \quad X_A = \frac{3}{8} \frac{Fl^2}{h(2h+3l)}$$

Diagramme de $M_3$

AB $\quad M_3 = -X_A y$

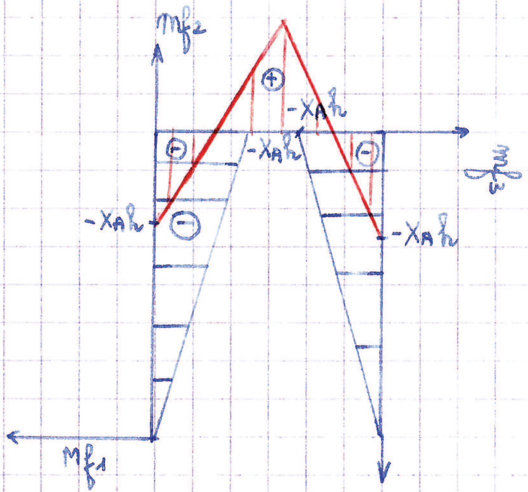
**BE** $M_3 = -[X_A h - Y_A x] = Y_A x - X_A h$

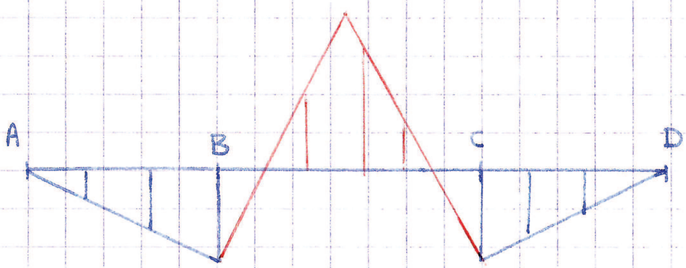
D'où $\begin{cases} M_3(h) = -X_A h \\ M_3(0) = -X_A h \\ M_3\left(\frac{\ell}{2}\right) = Y_A \frac{\ell}{2} - X_A h \end{cases}$

$\begin{cases} M_3 = 0 \Rightarrow x = \dfrac{X_A h}{Y_A} \\ x = \dfrac{3}{8} \dfrac{F\ell^2 h}{h(2h+3\ell)\frac{F}{2}} = \dfrac{3}{4} \dfrac{\ell^2}{(2h+3\ell)} \\ \text{cas } h = \ell \Rightarrow x = \dfrac{3}{4} \dfrac{\ell^2}{5\ell} = \dfrac{3}{20} \ell \end{cases}$

$h = \ell \rightarrow M_3\left(\frac{\ell}{2}\right) = \frac{7}{40} \ell$

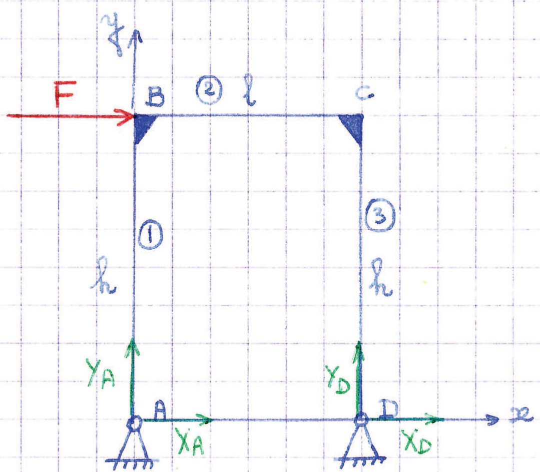
$X_A = \frac{3}{40} F$




structure étalée
(cas où $h = \ell$)

**EX 2**




1°) actions inconnues
- ménabréa
- 3 moments

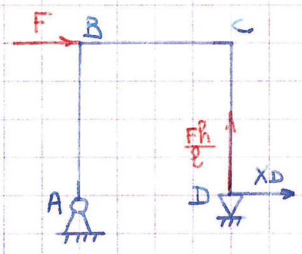
2°) diagramme de mf ($h = \ell$)

statique : $\begin{cases} X_A + X_D + F = 0 \\ Y_A + Y_D = 0 \\ -Fh + Y_D \ell = 0 \end{cases}$ $\Rightarrow$ $Y_D = \dfrac{Fh}{\ell} = -Y_A$

$\Rightarrow$ 2 inconnues et 1 équation

$H = 1$

équation supplémentaire.




$$\frac{\partial W(ABCD)}{\partial X_D} = 0 \qquad \left\{ X_D : \text{ inconnue hyperstatique} \right.$$

AB

$$M_3 = -F(h-y) + \frac{Fh}{\ell}\times\ell + y X_D = (F+X_D)y$$

$$\frac{\partial M_3}{\partial X_D} = y \qquad ds = dy \qquad (y \in (0,h))$$

BC

$$M_3 = h X_D + (\ell - x)\frac{Fh}{\ell} \qquad (ds = dx)$$

$$\frac{\partial M_3}{\partial X_D} = h \qquad (x \in (0,\ell))$$

CD

$$M_3 = y X_D$$

$$\frac{\partial M_3}{\partial X_D} = y \rightarrow ds = -dy \qquad y \in (h,0)$$

signe -

D'où $\displaystyle \frac{\partial W}{\partial X_D} = \frac{1}{EI}\int_{ABCD} M_3 \frac{\partial M_3}{\partial X_D}\, ds$

$$\Longrightarrow \int_0^h (F+X_D)\,y^2\,dy + \int_0^\ell \left(h^2 X_D + (\ell - x)\frac{Fh^2}{\ell}\right) dx + \int_h^0 X_D\, y^2 (-dy) = 0$$

$$[F+X_D]\frac{h^3}{3} + X_D h^2 \ell + \frac{Fh^2}{\ell}\left[-\frac{(\ell-x)^2}{2}\right]_0^\ell - \frac{X_D}{3}(-h^3) = 0$$

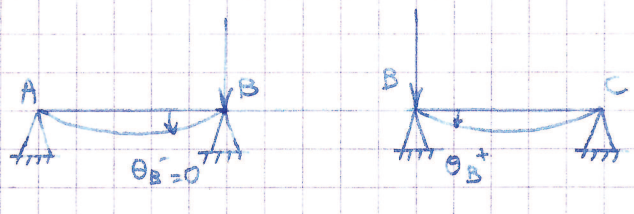
$$\Longrightarrow X_D\left(\frac{2h^3}{3} + h^2\ell + \frac{h^3}{3}\right) = -\frac{Fh^3}{3} - \frac{Fh^2}{\ell}\,\frac{\ell^2}{2} = -Fh^2\left(\frac{h}{3} + \frac{\ell}{2}\right)$$

$$\Longrightarrow X_D = -\frac{F}{2} < 0.$$

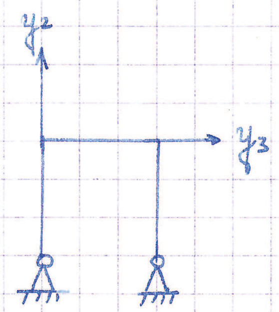
## 2ᵉ méthode : 3 moments

- travée ABC :

$$M_A h + 2M_B(h+l) + M_C l = 6EI\left(\theta_B^+ - \theta_B^-\right) + 6EI\left[\frac{Y_A - Y_B}{h} + \frac{Y_C - Y_B}{l}\right]$$




Y : déplacement.




AB $\begin{cases} Y_A = 0 \\ Y_B = -Y \end{cases}$ dplt horizontal.

BC $\begin{cases} Y_C = 0 \\ Y_B = 0 \end{cases}$ pas de dénivellation de B / C

$$2M_B(h+l) + M_C l = 6EI\frac{Y}{h} \quad (I)$$

travée BCD

$$M_B l + 2M_C(h+l) + \overset{0}{M_D} h = 0 + 6EI\left[\frac{\overset{0}{Y_B} - \overset{0}{Y_C}}{l} + \frac{\overset{+Y}{Y_D} - \overset{0}{Y_C}}{h}\right]$$

et $\quad M_B l + 2M_C(h+l) = -6EI\dfrac{Y}{h} \quad (II)$

résolution

$M_C = X_D h$ et $M_B = -X_A h$.

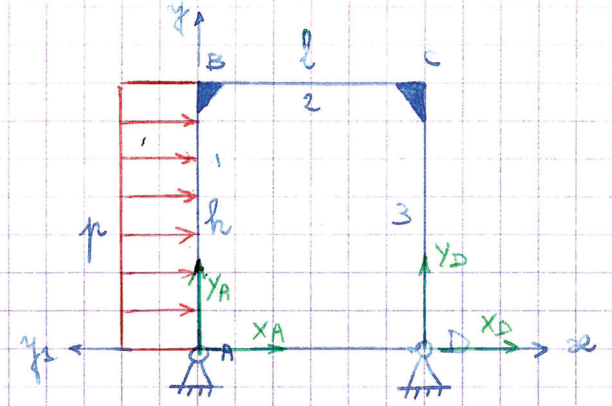
(I) et (II) $\Rightarrow$ $\begin{cases} -2X_A h(h+l) + X_D hl = 6EI\dfrac{Y}{h} \\ -X_A hl + 2X_D h(h+l) = -6EI\dfrac{Y}{h} \end{cases}$

$$\Rightarrow \quad -2X_A h(h+l) + X_D hl = X_A hl - 2X_D h(h+l)$$

$$X_A h\left[-2h - 3l\right] = -X_D h(2h+3l) \Rightarrow X_A = X_D$$

et $\quad Y = \dfrac{h}{6EI}\left[2\dfrac{F}{2}h(h+l) - \dfrac{F}{2}hl\right] = \dfrac{Fh^2}{6EI}\left(h + \dfrac{l}{2}\right) \qquad Y_B = -Y$

EX 1




1') Actions inconnues.

2') Allure déformée

méthode des 3 moments.

statique:

$$\begin{cases} X_A + X_D + ph = 0 \\ Y_A + Y_D = 0 \\ -\dfrac{ph^2}{2} + Y_D l = 0 \end{cases} \Rightarrow \underline{\underline{H = 1}}$$

équation supplémentaire

$M_A = 0$

$\theta_B^+ = 0$ — suivant $y_1$

<u>ABC</u>
$$2 M_B (h+l) + M_C l = 6EI\left(-\theta_B^-\right) + 6EI\left[\overbrace{\frac{y_A - y_B}{h}}^{AB} + \overbrace{\frac{y_C - y_B}{l}}^{BC}\right]$$

travée 2 — travée 3

$$y_B = -y \quad \text{et} \quad \theta_B^- = \frac{ph^3}{24EI}$$

$$2M_B(h+l) + M_C l = -6EI\,\frac{ph^3}{24EI} + 6EI\,\frac{y}{h} \qquad (1)$$

sur $y_2$ — sur $y_3$

<u>BCD</u>
$$M_B l + 2M_C(h+l) = 0 + 6EI\left[\frac{y_B - y_C}{l} + \frac{y_D - y_C}{h}\right]$$

$$M_B l + 2M_C(h+l) = -6EI\,\frac{y}{h}$$

or
$$\begin{cases} M_B = -\left(X_A h + \dfrac{ph^2}{2}\right) \\ M_C = X_D h \end{cases} \Rightarrow \begin{cases} -\left(X_A h + \dfrac{ph^2}{2}\right)l + 2X_D h(h+l) = -\dfrac{ph^3}{4} \ldots \\ -X_D h l + 2\left(X_A h + \dfrac{ph^2}{2}\right)(h+l) \;; \end{cases}$$

donc

$$X_A h\left[-l - 2(h+l)\right] + X_D h\left[2h + 2l + l\right] = \frac{hp}{2}\left(hl - \frac{h^2}{2} + 2(h+l)h\right)$$

$$\Rightarrow \quad -X_A(2h+3l) + X_D(2h+3l) = \frac{ph}{2}\left(l - \frac{h}{2} + 2h + 2l\right)$$

$$-X_A + X_D = \frac{ph}{2}\,\frac{3}{2}\,\frac{(2l+h)}{3l+2h} = \frac{3ph}{4}\,\frac{2l+h}{3l+2h} \;;$$

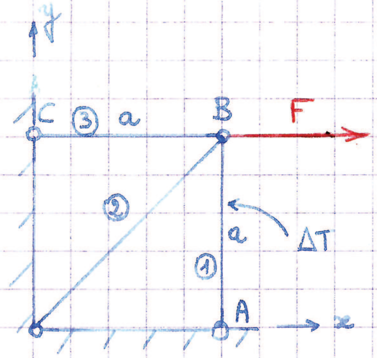
Avec équation de la statique.

$$\Rightarrow \quad 2X_D = ph\left[\frac{3(2l+h)}{4(3l+2h)} - 1\right] \Rightarrow X_D = -\frac{ph}{8}\left(\frac{6l+5h}{3l+2h}\right) < 0$$

supposons $h = \ell$

$\Rightarrow X_D = \frac{-11\, ph}{40} \Rightarrow X_A = \left(\frac{11}{40} - 1\right) ph = \frac{-29}{40}\, ph < 0.$

EX 1




état ⓪ : température uniforme $T_0$

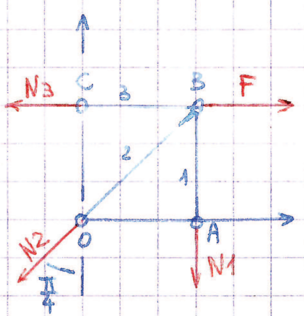
état ① : $\vec{F} = F\vec{x}$

barre 1 $\to T = T_0 + \Delta T$

1) Actions inconnues

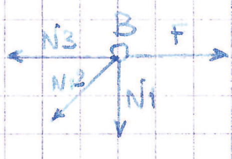
2) déplacement de B ?

étude de l'isostatisme : les barres sont tendues




$$\left.\begin{aligned} &\text{proj}/x : \quad F - N_3 - N_2 \frac{\sqrt{2}}{2} = 0 \\ &\text{proj}/y \quad -N_1 - N_2 \frac{\sqrt{2}}{2} = 0 \end{aligned}\right\} \quad H_e = 1$$

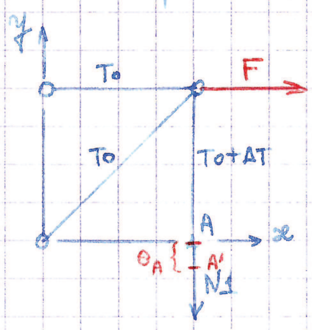
2° méthode isolons le noeud B




m̂ équation $H_i = 1$

b) équation supplémentaire

système isostatique équivalent




$\frac{\partial W}{\partial N_1} = 0$ avec $W = W_M + W_T \quad (1+2+3)$

mécanique ↗ ↑ thermique.

$$W_M = \frac{N_3^2 a}{2ES} + \frac{N_2^2 a\sqrt{2}}{2ES} + \frac{N_1^2 a}{2ES}$$

m̂ matériaux, m̂ section. (on remplace tt fct de $N_1$ : équ stat)

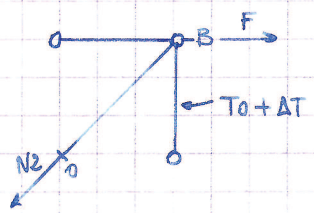
$$W_M = \frac{a}{2ES}\left[N_1^2 + N_1^2\, 2\sqrt{2} + (F + N_1)^2\right]$$

$W_T = -N1\,\vec{y}\cdot\vec{\theta_A}$ (le pt B ne bouge pas les barres 2 et 3 restent fixes.) 

$\vec{\theta_A} = -\alpha\, a\, \Delta T\, \vec{y}$

vu $l = l_0 + \alpha l_0 (T - T_0)$

si :



$\Rightarrow W_T = \vec{N2}\cdot\vec{\theta_0} + \vec{F}\cdot\vec{\theta_B}$.

dans notre cas $W_T = -N_1\vec{y}\cdot(-\alpha\, a\, \Delta T)\vec{y} = \alpha\, a\, N1\, \Delta T$

$\rightarrow W = \frac{a}{2ES}\left[N_1^2 + 2\sqrt{2}\,N_1^2 + (F+N_1)^2\right] + \alpha\, a\, \Delta T\, N1$

$\Rightarrow$

$$\frac{\partial W}{\partial N_1} = \frac{a}{ES}\left[N1 + 2\sqrt{2}\,N1 + F + N1\right] + \alpha a \Delta T = 0$$

$$\Longrightarrow 2N1(1+\sqrt{2}) = -\alpha ES \Delta T - F$$

$$\Longrightarrow N1 = \frac{-\alpha ES \Delta T + F}{2(1+\sqrt{2})} \quad (\text{barre / noeud})$$

et $N2 = -N1\sqrt{2}$ (statique) $\Longrightarrow N2 = \frac{\alpha ES \Delta T + F}{\sqrt{2}(1+\sqrt{2})}$ et N3.

si $F = 0$ $\quad \Delta T > 0$ échauffement

$N1 < 0$ comprimée

$N2 > 0$ tendue

$N3 < 0$ comprimée

si $\Delta T < 0$ refroidissement

$N1 > 0$ tendue

$N2 < 0$ comprimée

$N3 > 0$ tendue

déplacement de B ; $u_B$ = allongement de la barre 3.

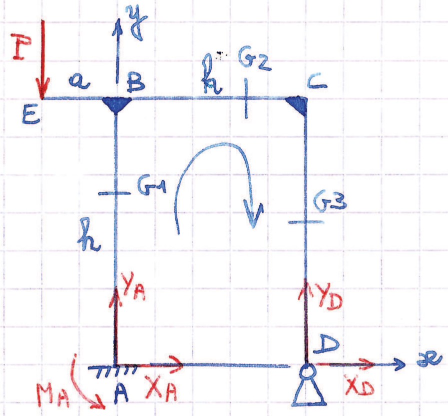
$u_B$ ; $\quad \frac{u_B}{a} = \frac{1}{E}\frac{(-N3)}{S} \Rightarrow u_B = -\frac{a}{ES}N3 = -\frac{a}{ES}(F+N1)$

$$u_B = -\frac{a}{ES}\left(F - \frac{\alpha ES \Delta T + F}{2(1+\sqrt{2})}\right) \Rightarrow u_B = \frac{a}{2ES(1+\sqrt{2})}\left[(1+2\sqrt{2})F - \alpha ES \Delta T\right]$$

pour $v_B$ barre 1 :

$\Rightarrow \quad \frac{v_B}{a} = \frac{1}{E}$

## EX 1

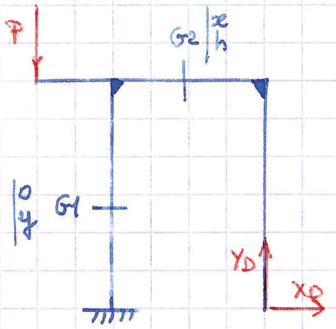



statique :

$$\begin{cases} X_A + X_D = 0 \\ Y_A + Y_D - P = 0 \quad \longrightarrow \quad H = 2 \\ M_A + Y_D l + aP = 0 \end{cases}$$

équation supplémentaire

Ménabréa




$\frac{\partial W}{\partial X_D} = 0$ et $\frac{\partial W}{\partial Y_D} = 0$

**G1 ∈ (AB)** $\qquad \frac{\partial M_3}{\partial X_D} = y \; ; \; \frac{\partial M_3}{\partial Y_D} = h \qquad ds = dy$

$M_3 = y X_D + h Y_D + aP.$

$W =$ potentiel $(AB + BC + CD + BE)$

**G2 ∈ (BC)** $\quad M_3 = X_D h + (h - x) Y_D \qquad \frac{\partial M_3}{\partial X_D} = h \qquad \frac{\partial M_3}{\partial Y_D} = h - x \qquad ds = dx$

**G3 ∈ (CD)**

$M_3 = X_D y \qquad \frac{\partial M_3}{\partial X_D} = y \qquad \frac{\partial M_3}{\partial Y_D} = 0 \qquad ds = -dy$

**G4 ∈ (BE)** $\quad M_3 = (a - x) P \qquad \frac{\partial W_{EB}}{\partial X_D} = 0$

1ère équation supplémentaire

$\frac{\partial W}{\partial X_D} = 0 \qquad W = \int_{ABCD} \frac{M_3^2}{2EI} ds$

$\Rightarrow \quad \frac{\partial W}{\partial \ldots} = \frac{1}{\ldots} \int \ldots M_3 \frac{\partial M_3}{\ldots} ds \; ;$

$$\int_0^h \left[X_D\, y + h\, Y_D + aP\right] h\,dy + \int_0^h \left[X_D h + (h-x) Y_D\right](h-x)\,dx = 0$$

$$\Longrightarrow \quad h\, X_D \frac{h^2}{2} + h^2 Y_D h + aP h^2 + X_D h \frac{h^2}{2} + Y_D \frac{h^3}{3} = 0$$

$$\Longrightarrow \quad X_D h + \frac{4}{3} Y_D h + aP = 0 \qquad (1)$$

2$^e$ équation $\quad \frac{\partial W}{\partial X_D} = 0 \iff \int_0^h (X_D\, y + h Y_D + aP)\, y\,dy + \int_0^h (X_D h + (h-x) Y_D)\, h\,dx$

$$+ \int_h^0 X_D\, y^2 (-dy) = 0$$

bornes

$$\iff \quad X_D \frac{h^3}{3} + h\, Y_D \frac{h^2}{2} + aP \frac{h^2}{2} + X_D h^2 h + Y_D h \frac{h^2}{2} + X_D \frac{h^3}{3} = 0\,;$$

$$\iff \quad X_D h \left(\frac{1}{3} + 1 + \frac{1}{3}\right) + h\, Y_D + \frac{aP}{2} = 0$$

$$\Longrightarrow \quad \frac{5}{3} h X_D + h Y_D + \frac{aP}{2} = 0\,; \qquad (2)$$

(1) et (2) $\rightarrow \quad Y_D\, h \left(\frac{11}{3}\right) + aP \frac{7}{2} = 0 \quad \Longrightarrow \quad Y_D = -\frac{21}{22} \frac{aP}{h} < 0\,;$

$$X_D = \frac{-aP}{h}\left(1 - \frac{21}{22}\,\frac{4}{3}\right) = \frac{3}{11}\frac{aP}{h}\,;$$

Avec bresse au pt A $\begin{cases} \omega_A = 0 \\ u_A = 0 \\ v_A = 0 \end{cases}$ au pt D $\begin{cases} u_D = 0 \\ v_D = 0 \end{cases}$

Déplacement du point B :

$$u_B = u_A - \omega_A (y_B - y_A) - \int_{AB} \frac{M_3}{EI_3}(y_B - y)\,ds$$

$$u_B = \frac{-1}{EI}\int_0^h \left[y\, X_D + h\, Y_D + aP\right](h-y)\,dy$$

$$u_B = -\frac{1}{EI}\left\{ X_D \left[\frac{h y^2}{2} - \frac{y^3}{3}\right]_0^h + \left(-\frac{(h-y)^2}{2}\right)_0^h (h Y_D + aP)\right\}$$

$$u_B = -\frac{1}{EI}\left\{ X_D h^3 \frac{1}{6} + h^2 \frac{aP}{2} + \frac{h^3}{2} Y_D \right\}$$

$$u_B = \frac{-aP h^2}{2EI}\left(\frac{1}{3}\cdot\frac{3}{11} - \frac{21}{22} + 1\right) = \frac{-3}{44}\frac{aP h^2}{EI}$$

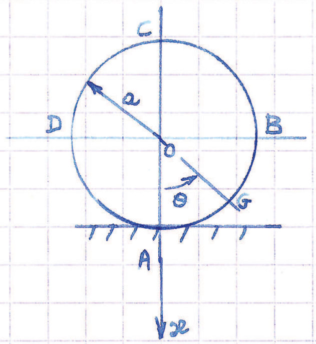
$$\omega_B = \omega_A + \int_{\overline{AB}} \frac{M_3}{EI_3}\, ds$$

$$= \frac{1}{EI} \int_0^h (y X_D + h Y_D + aP)\, dy = \frac{h}{EI}\left[X_D \frac{h^2}{2} + Y_D h^2 + aPh\right]$$

$$\omega_B = \frac{2}{11} \frac{haP}{EI}$$

sans pt d'inflexion avec pt d'inflexion

EX 1



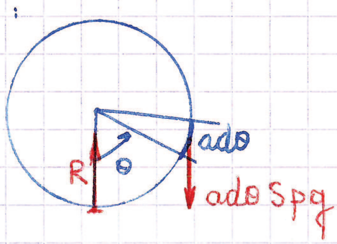

Anneau rayon a soumis à son propre poids, section constante S densité ρ

1) torseur des efforts intérieur (T, N, M)

2) déplacement pt C

$$\begin{cases} T = T(\theta) \\ N = N(\theta) \\ M = M(\theta) \end{cases}$$

d'où :

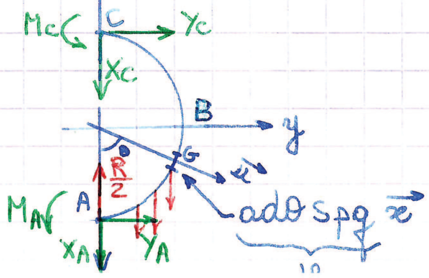



équilibre extérieur

$/\vec{x}$ : $-R + \underbrace{a\, 2\pi S \rho g}_{mg} = 0 \Rightarrow R = mg \qquad He = 0.$

hyperstaticité intérieure.

le seul axe possédant les 2 symétries (charg$^t$ et géométrie) = axe x




en A et C : algébriquement positif (d'office.)

statique :

$$\begin{cases} X_A + X_C \underbrace{- \frac{mg}{2}}_{\frac{R}{2}} + \frac{mg}{2} = 0 \\ Y_A + Y_C = 0 \\ M_A - 2a Y_C + M_C - mg \frac{a}{\pi} = 0 \end{cases}$$

mt des efforts répartiés

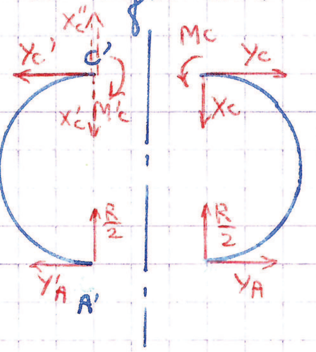
$$d\vec{mt} = \vec{AG} \wedge df\,\vec{x} = (-a\vec{x} + a\vec{u}) \wedge df\,\vec{x}$$

$$= a\,df\, \underbrace{\vec{u} \wedge \vec{x}}_{-\sin\theta\,\vec{z}} = -a\,df \sin\theta\,\vec{z} \quad (mt < 0)$$

$$\Rightarrow \vec{mt} = \int dm = \int_0^{\pi} -a\,a\,S\rho g\,d\theta \sin\theta\,\vec{z} = -\vec{z}\,a^2 S\rho g \int_0^{\pi} \sin\theta\,d\theta$$

$$\vec{mt} = -2a^2 S\rho g\,\vec{z} = \underbrace{-2\pi a S\rho}_{m} g\,\frac{a}{\pi}\,\vec{z} = -mg\frac{a}{\pi}\,\vec{z}$$

b) considérations de symétrie.




pour qu'il y ait symétrie et opposition des torseurs il faut $X_C = 0 = X_C' = X_C''$

$$\Rightarrow \begin{cases} Y_A + Y_C = 0 \\ M_A - 2a Y_C + M_C - mg\frac{a}{\pi} = 0 \end{cases} \quad \Rightarrow \quad H_i = 2 \quad (3 \text{ auparavant.})$$

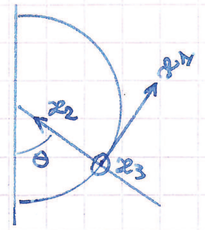
3) Equation suplementaire

Bresse.

$$A \begin{cases} \omega_A = 0 \\ u_A = 0 \\ v_A = 0 \end{cases} \qquad C \begin{cases} \omega_C = 0 \\ v_C = 0 \end{cases} \quad (\text{vu symétrie}).$$

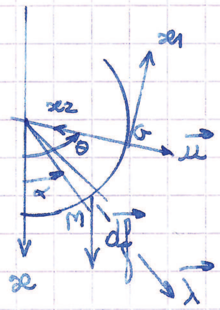
$$\underset{0}{\underset{\parallel}{\omega_C}} = \underset{0}{\underset{\parallel}{\omega_A}} + \int_{\widehat{AC}} \frac{M_3}{EI_3}\,ds \Rightarrow \int_{\widehat{AC}} \frac{M_3}{EI_3}\,ds = 0$$

$$\Rightarrow \int_{\widehat{AC}} M_3\,ds = 0$$




on passe de $\vec{x_1}$ à $\vec{x_2}$ par rotation $= +\frac{\pi}{2}$ (tjs)

$$/\vec{z} : \quad M_3 = -\left\{ M_A + a(1-\cos\theta)\, Y_A - a\sin\theta\, \frac{mg}{2} + \underbrace{\frac{1}{2}\frac{a}{\pi} mg\,(\theta\sin\theta + \cos\theta - 1)}_{\text{mt efforts répartis}} \right\}$$




$$\vec{dm} = \vec{GM} \wedge \vec{df}$$
$$= (-a\vec{u} + a\vec{\lambda}) \wedge df\, \vec{x}$$
$$= a\,df\,(-\vec{u}\wedge\vec{x} + \vec{\lambda}\wedge\vec{x})$$
$$= a\,df\,(\sin\theta\,\vec{z} - \sin\alpha\,\vec{z})$$
$$\vec{dm} = a(\sin\theta - \sin\alpha)\, a\, S\rho g\, d\alpha\, \vec{z}$$

$$\Rightarrow \quad \vec{mt} = a^2 S\rho g \int_0^\theta (\sin\theta - \sin\alpha)\, d\alpha$$
$$= a^2 S\rho g\, [\alpha\sin\theta + \cos\alpha]_0^\theta = a^2 S\rho g\, [\theta\sin\theta + \cos\theta - 1]$$

$$m = 2\pi a S\rho \quad \to \quad mt = \frac{1}{2}\frac{a}{\pi} mg\,(\theta\sin\theta + \cos\theta - 1)$$

$$\Leftrightarrow \quad a\int_0^\pi \left[M_A + aY_A - \frac{a\,mg}{2\pi}\right] d\theta + \int_0^\pi a\left[-aY_A + \frac{a\,mg}{2\pi}\right]\cos\theta\, d\theta$$
$$- \frac{amga}{2}\int_0^\pi \sin\theta\, d\theta + \frac{amga}{2\pi}\int_0^\pi \theta\sin\theta\, d\theta = 0$$

$$\Rightarrow \quad \left(M_A + aY_A - \frac{amg}{2\pi}\right)\pi + \frac{amg}{2}\underbrace{[\cos\theta]_0^\pi}_{-2} + \frac{amg}{2\pi}\left\{[\theta\cdot(-\cos\theta)]_0^\pi - \int_0^\pi -\cos\theta\, d\theta\right.$$
$$= 0$$

$$\Leftrightarrow \quad \pi M_A + a\pi Y_A - \frac{amg}{2} - amg + \frac{amg}{2} = 0$$

$$\Rightarrow \quad M_A + aY_A - \frac{amg}{\pi} = 0$$

2 éq

$$\underset{\overset{\|}{0}}{v_C} = \underset{\overset{\|}{0}}{v_A} + \underset{\overset{\|}{0}}{\omega_A}(x_C - x_A) + \int_{AC} \frac{M_3}{EI_3}(x_C - x)\, ds$$

$$\Rightarrow \int_{\widehat{AC}} M_3 \,(-a \underbrace{-a\cos\theta}_{x}) \, a\,d\theta = 0$$

$$\Rightarrow \int_{\widehat{AC}} M_3 \,(1+\cos\theta)\, d\theta = 0$$

$$\int_{\widehat{AC}} M_3 \, d\theta + \int_{\widehat{AC}} M_3 \cos\theta \, d\theta = 0$$

$\Rightarrow$ équation supplémentaire + résolution.